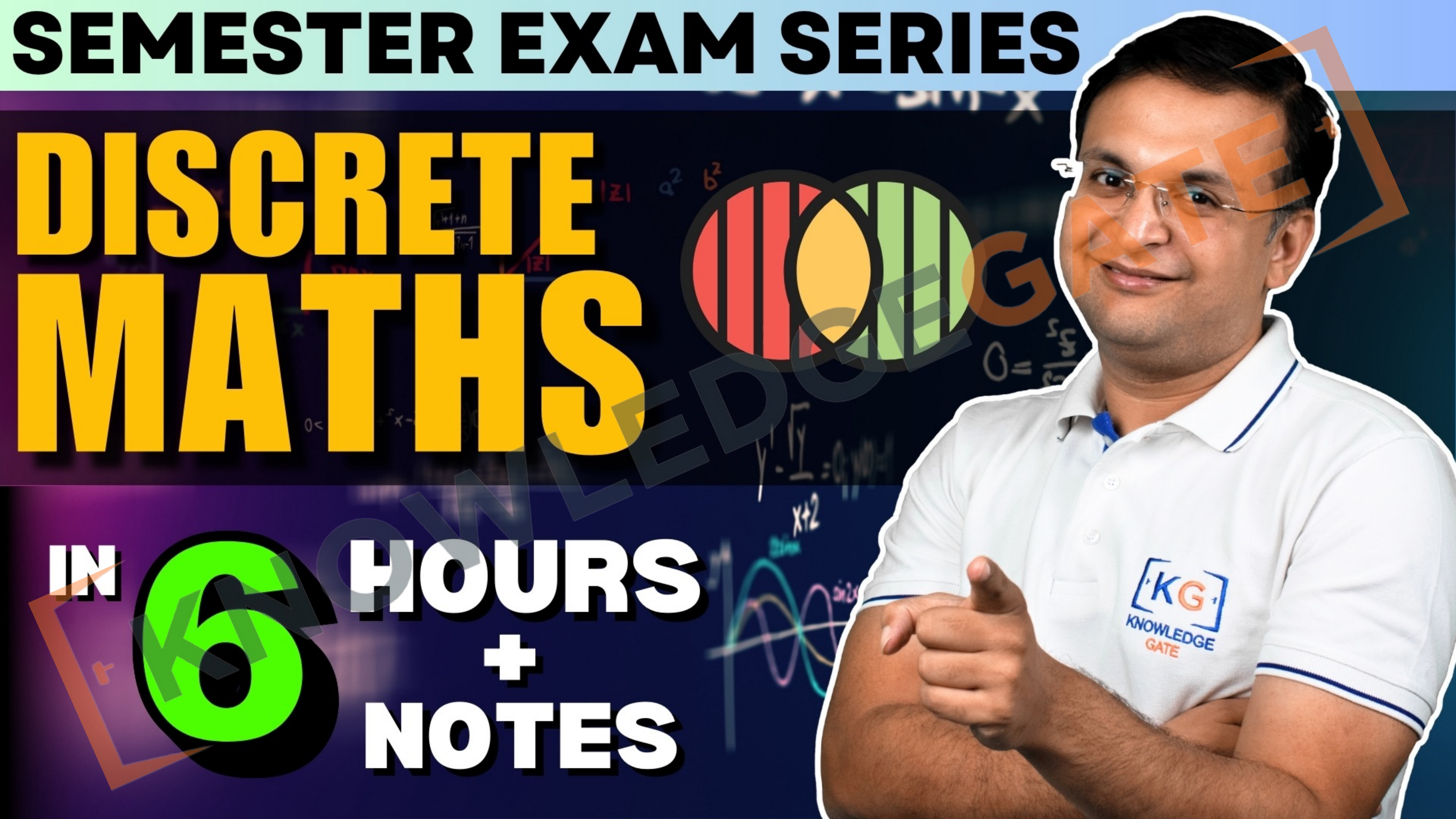
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Video chapters

- Chapter-1 (Set Theory):
- Chapter-2 (Relations):
- Chapter-3 (POSET & Lattices):
- Chapter-4 (Functions):
- Chapter-5 (Theory of Logics):
- Chapter-6 (Algebraic Structures):
- Chapter-7 (Graphs):
- Chapter-8 (Combinatorics):

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Chapter-1 (Set Theory)

What is a SET

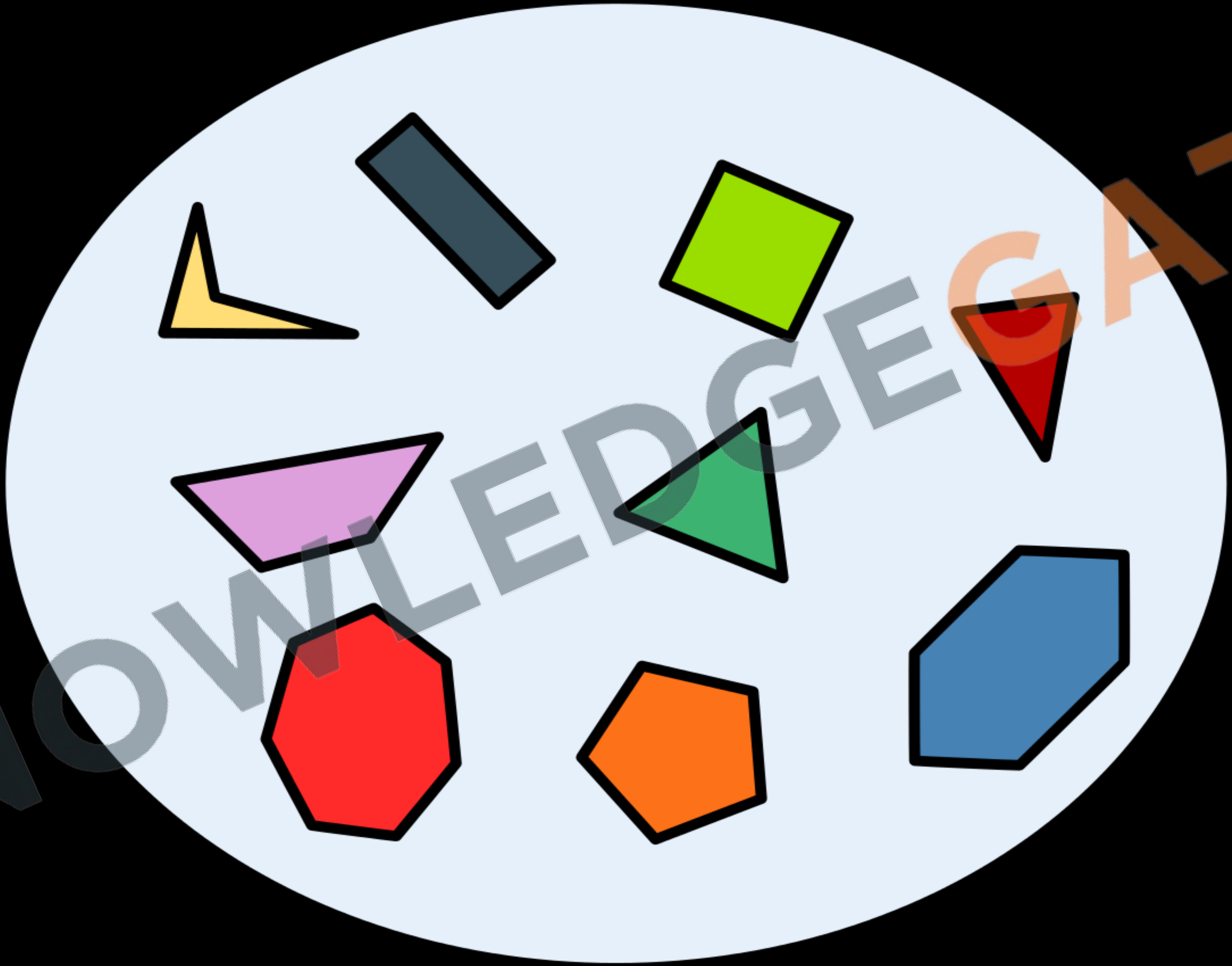
- Set are the fundamental discrete structures on which all the discrete structures are built. Sets are used to group objects together, formally speaking

- “An *unordered ,well-defined*, collection of *distinct* objects (Called elements or members of a set) of same type”. Here the type is defined by the one who is defining the set. For e.g.

- A = {0,2,4,6, ---}

- B = {1,3,5, ---}

- C= {x| x ∈ Natural number}



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- A Set is generally denoted usually by capital letter. The objects of a set called the elements, or members of the set.
- A set is said to contain its elements.
- Lower case letters are generally used to denote the elements of the set.

- $x \in A$, means element x is a member of A

- $x \notin A$ means x is not a member of A

- **Cardinality of a set** — It is the number of elements present in a Set, denoted like $|A|$.

- For e.g. $A = \{0,2,4,6\}$, $|A| = 4$

Representation of set

- **Tabular/Roster representation of set** - here a set is defined by actually listing its members. E.g.

- A = {a, e, i, o, u}

- B = {1, 2, 3, 4}

- C = {---- -4, -2, 0, 2, 4, -----}.

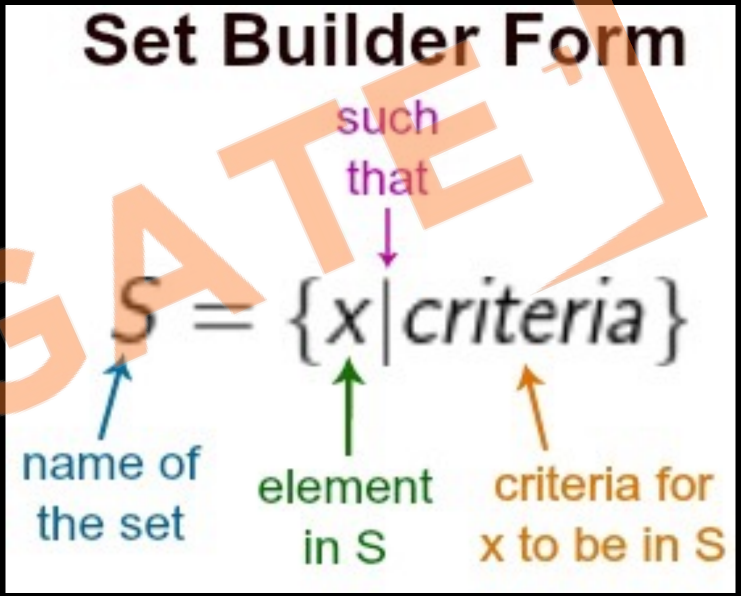
- **Set Builder representations of set**- here we specify the property which the elements of the set must satisfy. E.g.

- A = {x| x is an odd positive number less than 10},

- A = {x | x ϵ English alphabet && x is vowel}

- B = {x | x ϵ N && x <5}

- C = {x | x ϵ Z && x%2 = 0 }



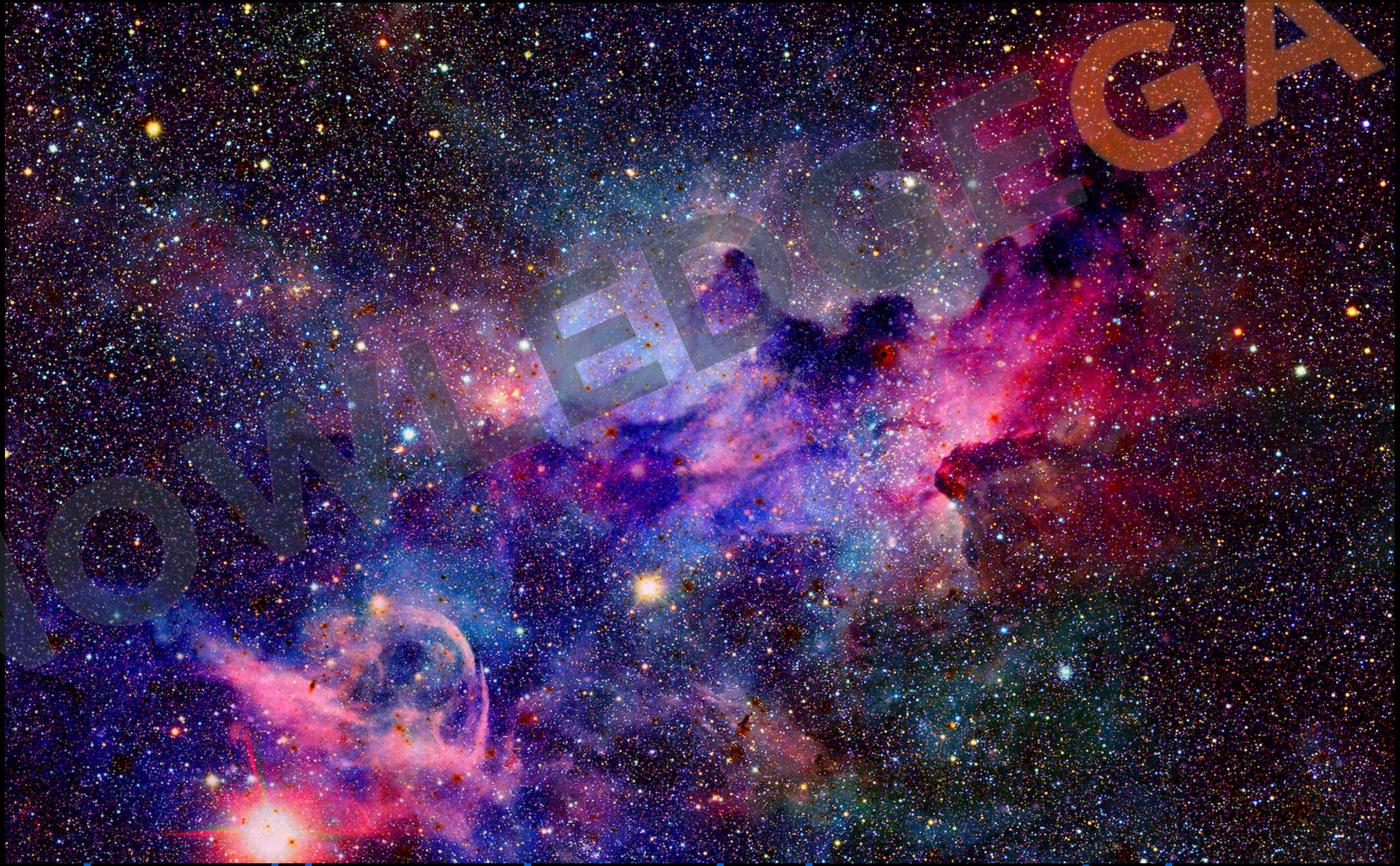
- **Finite set** - If there are exactly *'n'* elements in *S* where *'n'* is a nonnegative integer, we say that *S* is a *finite set.*

- *i.e. if a* set contain specific or finite number of elements is called is called finite set. For e.g. A = {1,2,3,4}

- **Infinite set** – A set contain infinite number of elements is called infinite set, if the counting of different elements of the set does not come to an end. For e.g. a set of natural numbers.

- **Null set / empty set** - Is the unique set having no elements. its size or cardinality is zero i.e. $|\phi| = 0$. It is denoted by a symbol ϕ or {}. A set with one element is called singleton set.

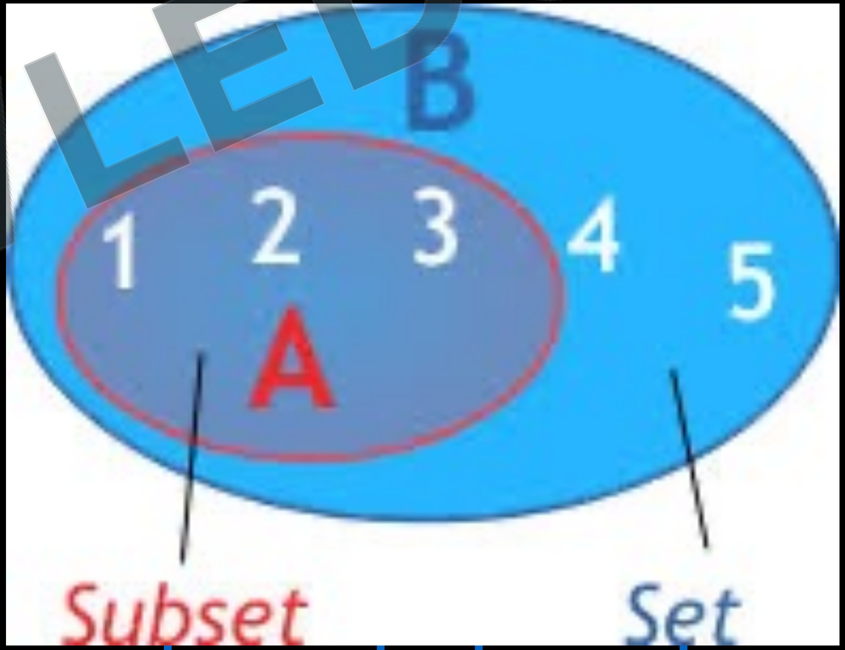
- **Universal set** – if all the sets under investigation are subsets of a fixed set, i.e. the set containing all objects under investigation, in Venn diagram it is represented by a rectangle, and it is denoted by U.



- **Subset of a set** – If every element of set A is also an element of set B i.e.
- $\forall x(x \in A \rightarrow x \in B)$, then A is called subset of B and is written as $A \sqsubseteq B$. B is called the superset of A.

- E.g. A = {1,2,3} B = {1,2,3,4,5}

- Note that to show that *A* is not a subset of *B* we need only find one element $x \in A$ with $x \notin B$. To show that $A \sqsubseteq B$, show that if $x \in A$, then $x \in B$.



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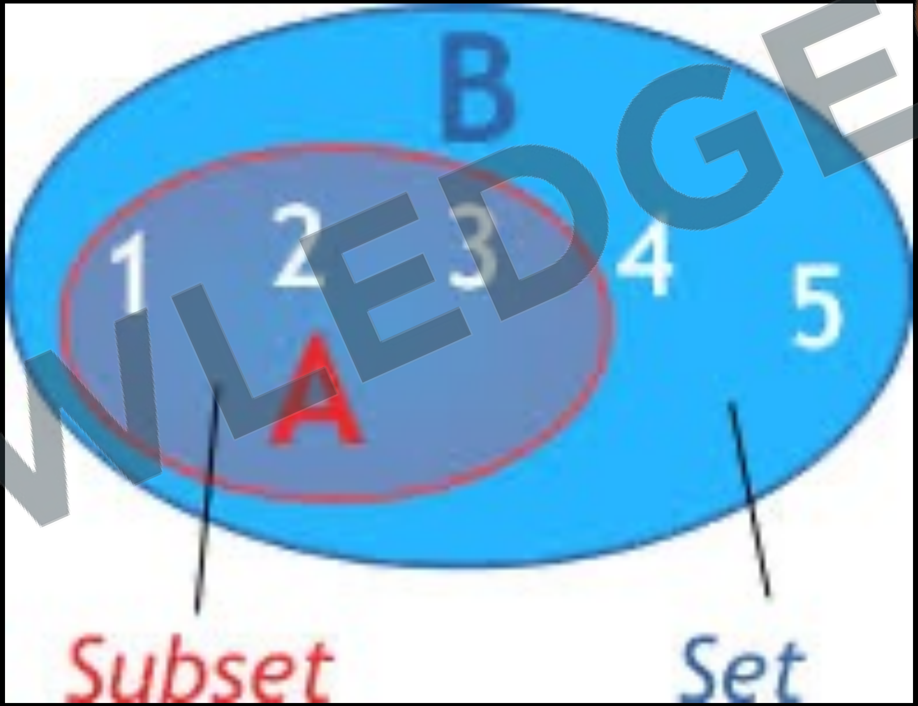
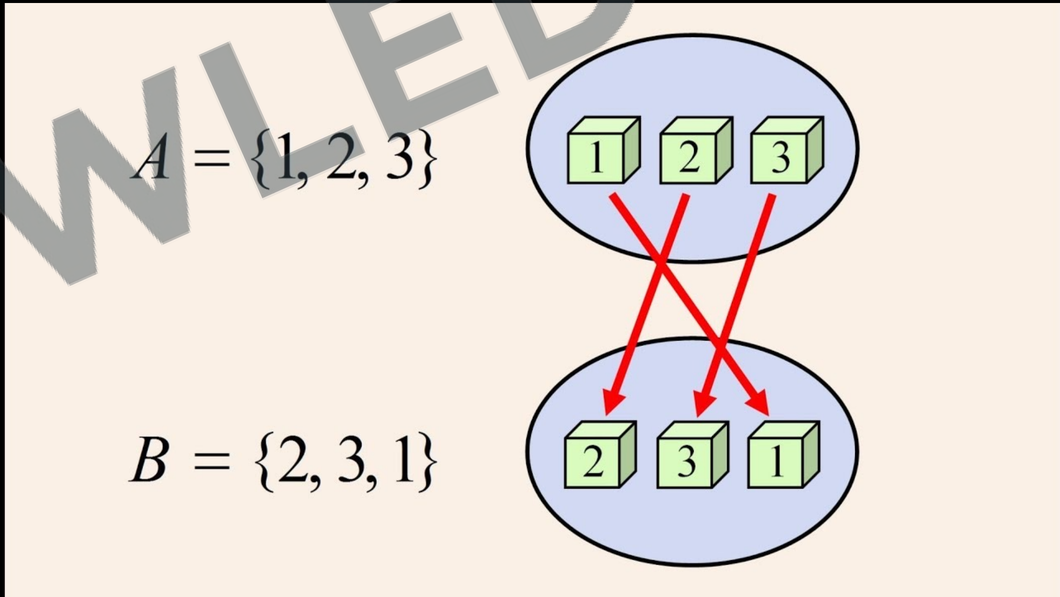
- $\phi \sqsubseteq A$, Empty Set ϕ is a subset for every set
- $A \sqsubseteq U$, Every Set is a subset of Universal set U
- $A \sqsubseteq A$, Every Set is a subset of itself.

- **Proper subset** – if A is a subset of B and A ≠ B, then A is said to be a proper subset of B, i.e. there is at least one element in B which is not in A. denoted as $A \subset B$.



- **Equality of sets** – If two sets A and B have the same element and therefore every element of A also belong to B and every element of B also belong to A, then the set A and B are said to be equal and written as A=B.
- if $A \sqsubseteq B$ and $B \sqsubseteq A$, then A=B
- $\forall x(x \in A \leftrightarrow x \in B)$



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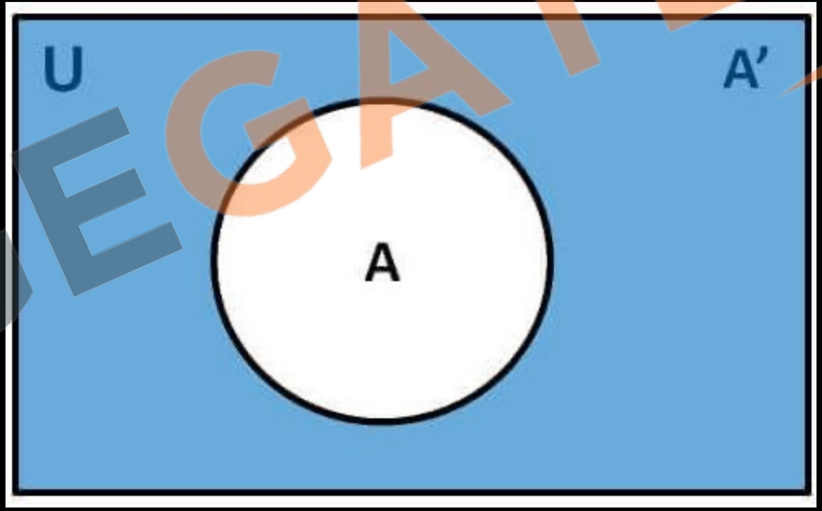
- **Power set** – let A be any set, then the set of all subsets of A is called power set of A and it is denoted by P(A) or 2^A.
- If A= {1,2,3}, then P(A) = {ϕ, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3}}
- Cardinality of the power set of A is n, $|P(A)| = 2^n$

Operation on sets

- **Complement of set** – Set of all x such that $x \notin A$, but $x \in U$.
- $A^c = \{x \mid x \notin A \;\&\; x \in U\}$

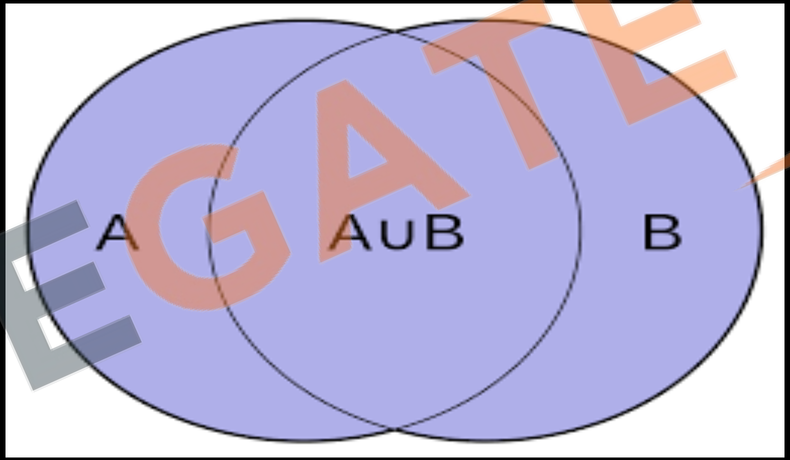


$U = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 3, 6\}$

$A^c = \{\quad\quad\}$

- **Union of sets** – Union of two sets A and B is a set of all those elements which either belong to A or B or both, it is denoted by A U B.
- A U B = {x| x ϵ A or x ϵ B}



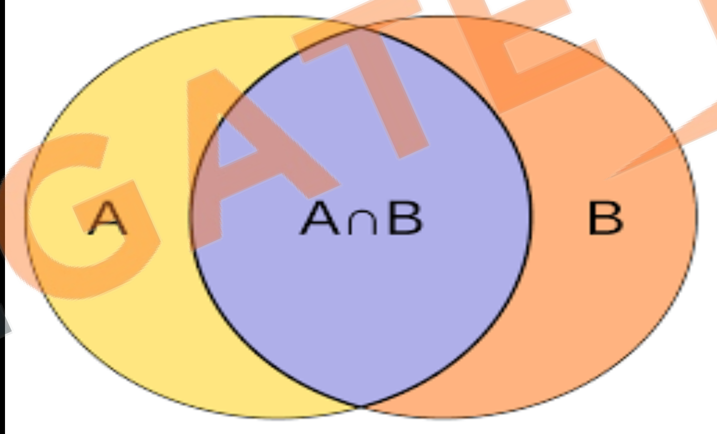
A = {1, 2, 3, 4}

B = {3, 4, 5, 6}

A U B = {   }

|A|+|B|=| A U B | ?

- **Intersection of sets** - Intersection of two sets A and B is a set of all those elements which belong to both A and B, and is denoted by $A \cap B$.
- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

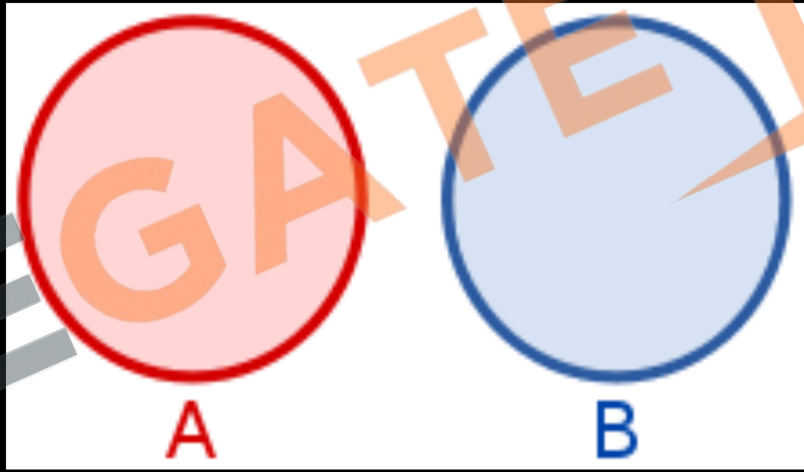


$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

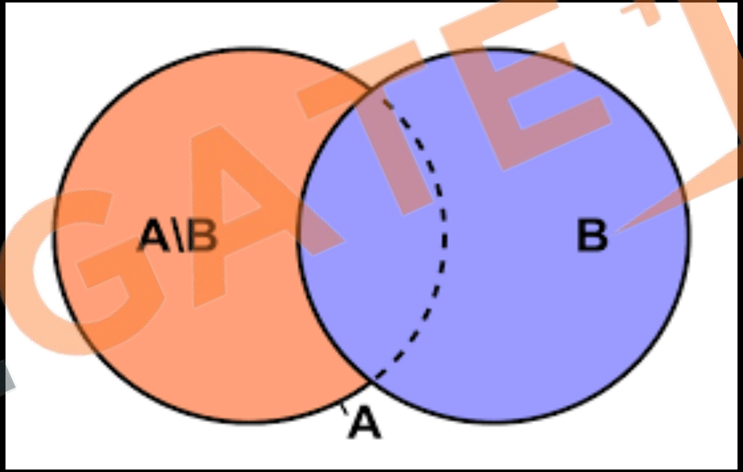
$A \cap B = \{\quad\quad\}$

- **Disjoint sets** -- Two sets are said to be disjoint if they do not have a common element, i.e. no element in A is in B and no element in B is in A.



- $A \cap B = \phi$

- **Set difference** – the set difference of two sets A and B, is the set of all the elements which belongs to A but do not belong to B.
- $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

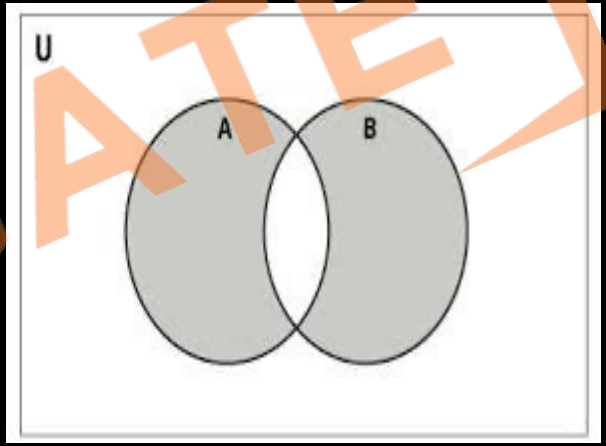


$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

$A - B = \{ \quad \}$

Symmetric difference – the symmetric difference of two sets A and B is the set of all the elements that are in A or in B but not in both, denoted as. $A \oplus B = (A \cup B) - (A \cap B)$



$A \oplus B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$

$A \oplus B = (A - B) \cup (B - A)$

$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

$A \oplus B = \{ \qquad\qquad \}$

Idempotent law

- $A \cup A = A$
- $A \cap A = A$

Associative law

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C= A \cap (B \cap C)$

Commutative law

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Distributive law

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's law

- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$

Identity law

- $A \cup \phi = A$
- $A \cap \phi = \phi$
- $A \cup U = U$
- $A \cap U = A$

Complement law

- $A \cup A^C = U$
- $A \cap A^C = \phi$
- $U^C = \phi$
- $\phi^C = U$

Involution law

- $((A)^C)^C = A$

Chapter-2 (Relations)

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Cartesian Product

1. Cartesian Product of two sets A and B in the set of all ordered pairs, whose first member belongs to the first set and second member belongs to the second set, denoted by $A \times B$.

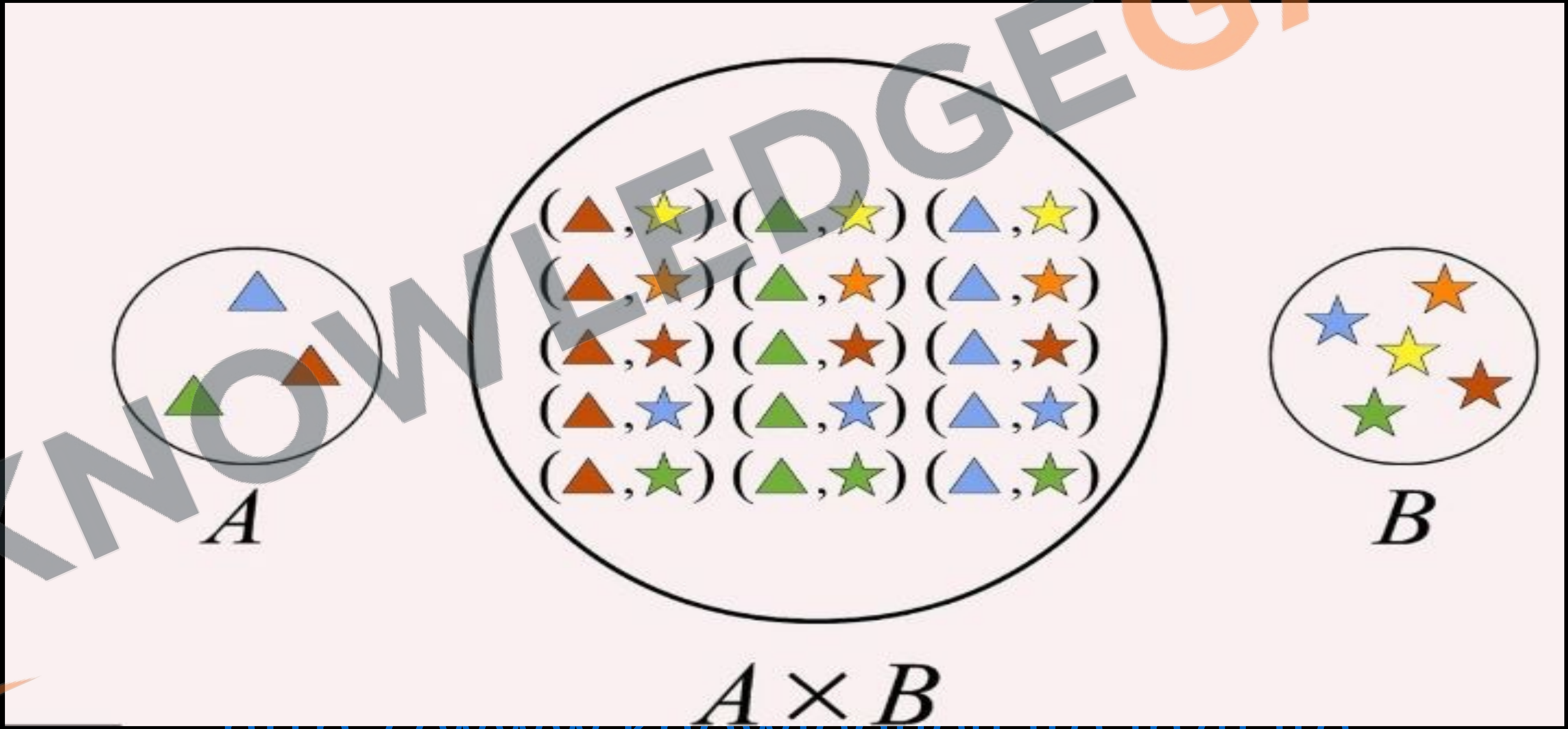
2. For E.g. if $A = \{a, b\}$, $B = \{1, 2, 3\}$

3. $A \times B = \{ \quad \}$

a
b

1
2
3

- It is a kind of maximum relation possible, where every member of the first set belong to every member of the second set.
- $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$



1. In general, commutative law does not hold good $A \times B \neq B \times A$

2. If $|A| = m$ and $|B| = n$ then $|A \times B| =$

Relation

- **Relation**: - Let A and B are sets then every possible subset of ‘A×B’ is called a relation from A to B.
- If |A| =m and |B| = n then total no of element(pair) will be m*n, every element will have two choice weather to present or not present in the subset(relation), therefore the total number of relation possible is ________

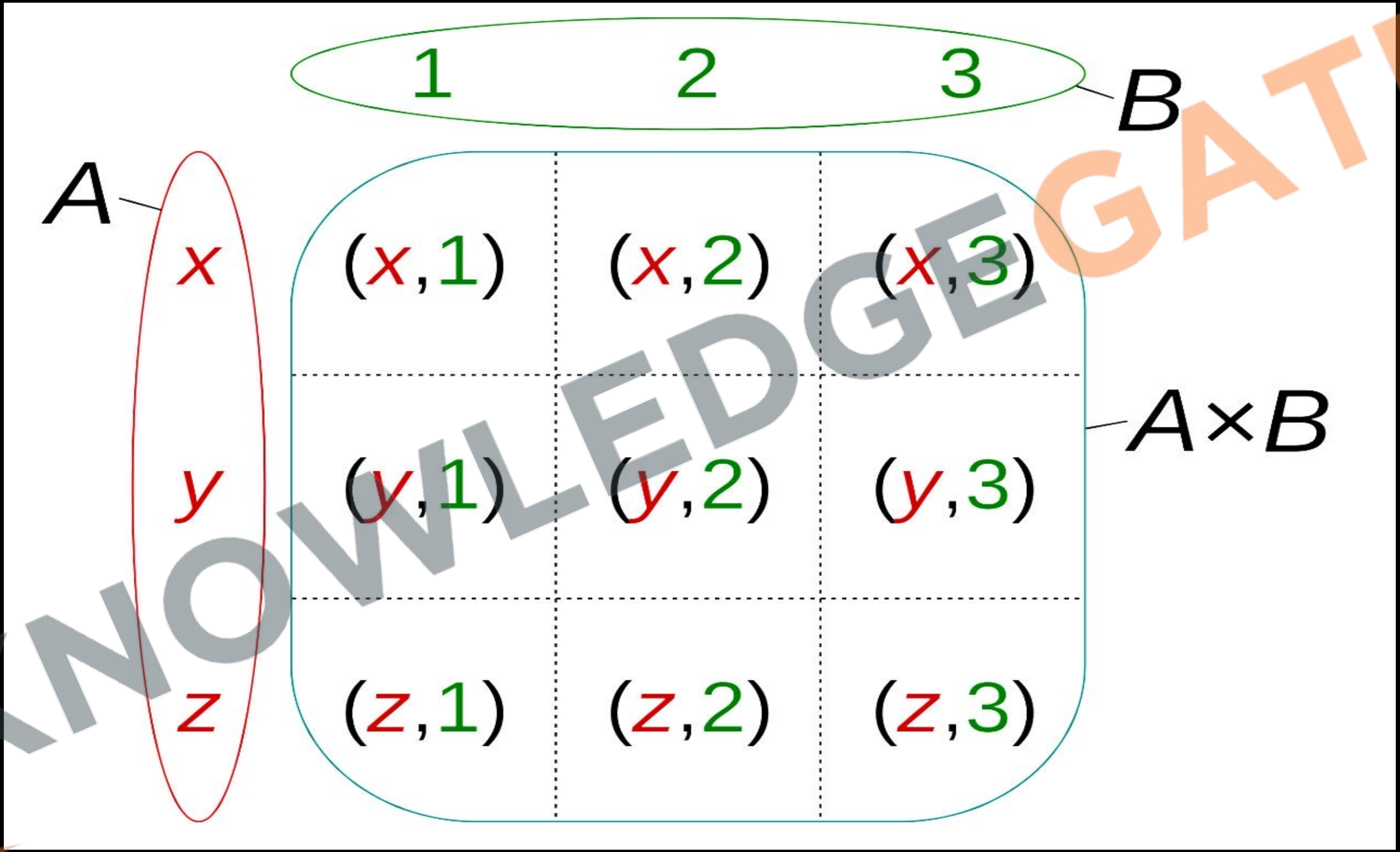
For E.g. if A = {a, b}, B = {1, 2}, A×B = {(a, 1), (a, 2), (b, 1), (b, 2)}

(a , 1)	(a , 2)	(b , 1)	(b , 2)	
0	0	0	0	
0	0	0	1	
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	

For E.g. if A = {a, b}, B = {1, 2}

(a , 1)	(a , 2)	(b , 1)	(b , 2)	
0	0	0	0	{ }
0	0	0	1	{(b, 2)}
0	0	1	0	{(b, 1)}
0	0	1	1	{(b, 1), (b, 2)}
0	1	0	0	{(a, 2)}
0	1	0	1	{(a, 2), (b, 2)}
0	1	1	0	{(a, 2), (b, 1)}
0	1	1	1	{(a, 2), (b, 1), (b, 2)}
1	0	0	0	{(a, 1)}
1	0	0	1	{(a, 1), (b, 2)}
1	0	1	0	{(a, 1), (b, 1)}
1	0	1	1	{(a, 1), (b, 1), (b, 2)}
1	1	0	0	{(a, 1), (a, 2)}
1	1	0	1	{(a, 1), (a, 2), (b, 2)}
1	1	1	0	{(a, 1), (a, 2), (b, 1)}
1	1	1	1	{(a, 1), (a, 2), (b, 1), (b, 2)}

Matrix Representation



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- **<u>Complement of a relation</u>**: - Let R be a relation from A to B, then the complement of relation will be denoted by R’, R^C or ~~R.~~

- $R' = \{(a, b) \mid (a, b) \in A\times B, (a, b) \in! R\}$

- $R' = (A\times B) - R$

- For E.g. if $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

- $R = \{(a, 1), (a, 3), (b, 2)\}$

- $R' = \{ \qquad \}$

$R \cup R' =$

$R \cap R' =$

- **Inverse of a relation**: - Let R be a relation from A to B, then the inverse of relation will be a relation from B to A, denoted by R^{-1}.

- $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

- $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

- $R = \{(a, 1), (a, 3), (b, 2)\}$

- $R^{-1} = \{ \qquad \qquad \qquad \}$

- $|R| \; \square \; |R^{-1}|$

- **Diagonal relation**: - A relation R on a set A is said to be diagonal relation if, R is a set of all ordered pair (x, x), for every ∀x ∈ A, sometimes it is also denoted by $\blacktriangle_A$

- R = {(x, x) | ∀ x ∈ A}

	1	**2**	**3**
1	11		
2		22	
3			33

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Types of a Relation

- To further study types of relations, we consider a set A with n elements, then a cartesian product A×A will have n^2 elements(pairs). Therefore, total number of relation possible is $2^{n * n}$

- **Reflexive relation**: - A relation R on a set A is said to be reflexive,
- If $\forall x \in A$
  - $(x, x) \in R$

	1	2	3
1	11		
2		22	
3			33

Q consider a set A = {1,2,3}, find which of the following relations are reflexive and Irreflexive?

	Relation	Reflexive	Irreflexive
1	$A \times A$		
2	ϕ		
3	{(1,1), (2,2), (3,3)}		
4	{(1,2), (2,3), (1,3)}		
5	{(1,1), (1,2), (2,1), (2,2)}		
6	{(1,1), (2,2), (3,3), (1,3), (2,1)}		
7	{(1,3), (2,1), (2,3), (3,2)}		

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	{(a, b), (b, a), (b, b)}
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	{(a, a), (a, b), (b, a)}
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	
0	0	0	1	
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

- **Irreflexive relation**: - A relation R on a set A is said to be Irreflexive,

1. If $\forall x \in A$
2. $(x, x) \notin R$

Q consider a set A = {1,2,3}, find which of the following relations are reflexive and Irreflexive?

	Relation	Reflexive	Irreflexive
1	$A \times A$	Y	
2	ϕ	N	
3	{(1,1), (2,2), (3,3)}	Y	
4	{(1,2), (2,3), (1,3)}	N	
5	{(1,1), (1,2), (2,1), (2,2)}	N	
6	{(1,1), (2,2), (3,3), (1,3), (2,1)}	Y	
7	{(1,3), (2,1), (2,3), (3,2)}	N	

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	{(a, b), (b, a), (b, b)}
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	{(a, a), (a, b), (b, a)}
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	
0	0	1	0	{(b, a)}
0	0	1	1	
0	1	0	0	{(a, b)}
0	1	0	1	
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	

- **Symmetric relation**: - A relation R on a set A is said to be Symmetric,

If $\forall a, b \in A$

$(a, b) \in R$

………………………………

then $(b, a) \in R$

………………………………

Q consider a set A = {1,2,3}, find which of the following relations are Symmetric, Anti-Symmetric and Asymmetric?

	Relation	Symmetric	Anti-Symmetric	Asymmetric
1	$A \times A$			
2	ϕ			
3	{(1,1), (2,2), (3,3)}			
4	{(1,2), (2,3), (1,3)}			
5	{(1,1), (1,2), (2,1), (2,2)}			
6	{(1,1), (2,2), (3,3), (1,3), (2,1)}			
7	{(1,3), (2,1), (2,3), (3,2)}			

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	{(a, b), (b, a), (b, b)}
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	{(a, a), (a, b), (b, a)}
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	{(a, b), (b, a), (b, b)}
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	{(a, a), (a, b), (b, a)}
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

- **Anti-Symmetric relation**: - A relation R on a set A with cartesian product A×A is said to be Anti-Symmetric,

If $\forall$ a, b $\in$ A

(a, b) $\in$ R

(b, a) $\in$ R

………………………………

a = b

………………………………

Conclusion: Symmetry is not allowed but diagonal pairs are allowed

	Relation	Symmetric	Anti-Symmetric	Asymmetric
1	$A \times A$	Y		
2	ϕ	Y		
3	{(1,1), (2,2), (3,3)}	Y		
4	{(1,2), (2,3), (1,3)}	N		
5	{(1,1), (1,2), (2,1), (2,2)}	Y		
6	{(1,1), (2,2), (3,3), (1,3), (2,1)}	N		
7	{(1,3), (2,1), (2,3), (3,2)}	N		

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	{(a, b), (b, a), (b, b)}
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	{(a, a), (a, b), (b, a)}
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	
0	1	1	1	
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	
1	1	1	1	

- **Asymmetric relation**: - A relation R on a set A is said to be Asymmetric,

If $\forall$ a, b $\in$ A

(a, b) $\in$ R

……………………………

(b, a) $\notin$ R

……………………………

Conclusion: Symmetry is not allowed; even diagonal pairs are not allowed

	Relation	Symmetric	Anti-Symmetric	Asymmetric
1	A×A	Y	N	
2	ϕ	Y	Y	
3	{(1,1), (2,2), (3,3)}	Y	Y	
4	{(1,2), (2,3), (1,3)}	N	Y	
5	{(1,1), (1,2), (2,1), (2,2)}	Y	N	
6	{(1,1), (2,2), (3,3), (1,3), (2,1)}	N	Y	
7	{(1,3), (2,1), (2,3), (3,2)}	N	N	

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	{(a, b), (b, a), (b, b)}
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	{(a, a), (a, b), (b, a)}
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	
0	0	1	0	{(b, a)}
0	0	1	1	
0	1	0	0	{(a, b)}
0	1	0	1	
0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	

- **Transitive relation**: - A relation R on a set A is said to be Transitive,

If $\forall$ a, b, c $\in$ A

(a, b) $\in$ R

(b, c) $\in$ R

……………………………

(a, c) $\in$ R

……………………………

	Relation	Transitive
1	A×A	
2	ϕ	
3	{(1,1), (2,2), (3,3)}	
4	{(1,2), (2,3), (1,3)}	
5	{(1,1), (1,2), (2,1), (2,2)}	
6	{(1,1), (2,2), (3,3), (1,3), (2,1)}	
7	{(1,3), (2,1), (2,3), (3,2)}	
8	{(1,2)}	
9	{(1,3), (2,3)}	
10	{(1,2), (1,3)}	
11	{(2,3), (1,2)}	

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	{(a, b), (b, a)}
0	1	1	1	{(a, b), (b, a), (b, b)}
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	{(a, a), (a, b), (b, a)}
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

For E.g. if A = {a, b}, A×A = {(a, a), (a, b), (b, a), (b, b)}

(a , a)	(a , b)	(b , a)	(b , b)	
0	0	0	0	{ }
0	0	0	1	{(b, b)}
0	0	1	0	{(b, a)}
0	0	1	1	{(b, a), (b, b)}
0	1	0	0	{(a, b)}
0	1	0	1	{(a, b), (b, b)}
0	1	1	0	
0	1	1	1	
1	0	0	0	{(a, a)}
1	0	0	1	{(a, a), (b, b)}
1	0	1	0	{(a, a), (b, a)}
1	0	1	1	{(a, a), (b, a), (b, b)}
1	1	0	0	{(a, a), (a, b)}
1	1	0	1	{(a, a), (a, b), (b, b)}
1	1	1	0	
1	1	1	1	{(a, a), (a, b), (b, a), (b, b)}

- **Equivalence Relation**: - A relation R on a set A with cartesian product A×A is said to be Equivalence, if it is

1. **Reflexive**
2. **Symmetric**
3. **Transitive**

- If two relations R_1 and R_2 are Equivalence then their union need not to be equivalence but intersection will also be Equivalence.

R_1 : (a, b) iff (a + b) is even over the set of integers

R_2 : (a, b) iff (a + b) is odd over the set of integers

R_3 : (a, b) iff a x b > 0 over the set of non-zero rational numbers

R_4 : (a, b) iff $|a - b| \le 2$ over the set of natural numbers

- **Partial Order Relation**: - A relation R on a set A with cartesian product A×A is said to be partial order, if it is

1. **Reflexive**
2. **Anti - Symmetric**
3. **Transitive**

- **<u>Partial ordering set (Poset):</u>** - a set A with partial ordering relation R defined on A is called a POSET and is denoted by [A, R]

- For e.g. [A, /], [A, <=], [P(S), $\sqsubseteq$]

Q let R_1 be a relation from A = {1,3,5,7} to B = {2,4,6,8} and R_2 be another relation from B to C = {1,2,3,4} as defined below

(i) an element x in A is related to an element y in B if x + y is divisible by 3

(ii) an element x in B is related to an element y in C if x + y is even but not divisible by 3. Which is the composite relation R_1R_2 from A to C?

a) {(1,2), (1,4), (3,3), (5,4), (7,3)}

b) {(1,2), (1,3), (3,2), (5,2), (7,3)}

c) {(1,2), (3,2), (3,4), (5,4), (7,2)}

d) {(3,2), (3,4), (5,1), (5,3), (7,1)}

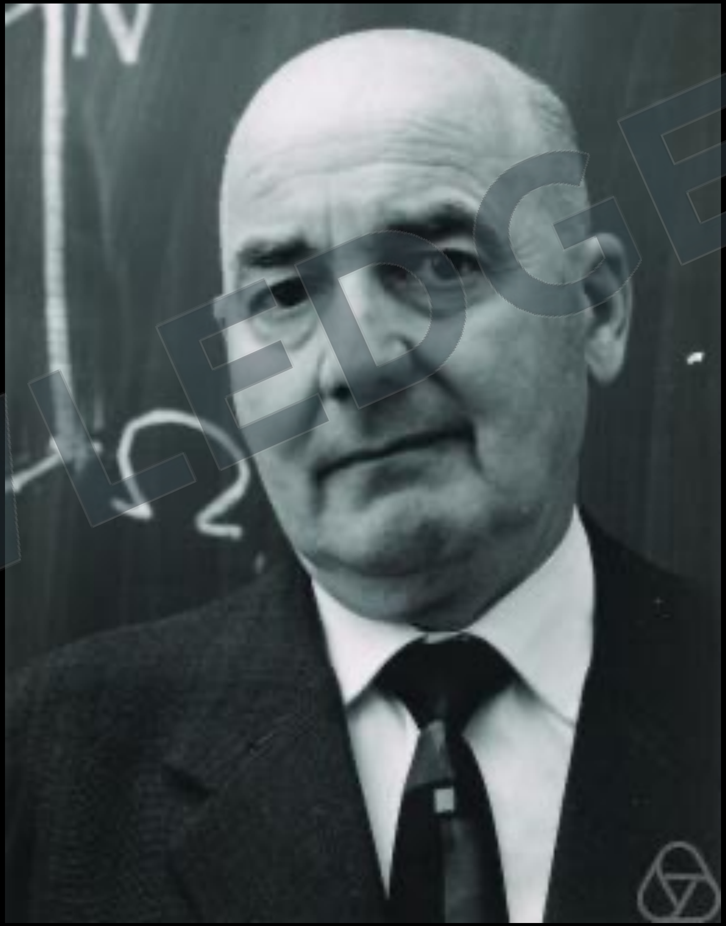
Chapter-3 (POSET & Lattices)

http://www.knowledgegate.in/gate

Conversion of POSET into a Hasse Diagram

- If we want to study Partial order relation further then it will be better to convert it into more convenient notation so that it can be studied easily.
- This graphical representation is called Hasse Diagram

- In order theory, a Hasse diagram is a type of mathematical diagram used to represent a finite partially ordered set, in the form of a drawing of its transitive reduction. The diagrams are named after Helmut Hasse (1898–1979)



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Steps to convert partial order relation into hasse diagram

1- Draw a vertex for each element in the Set

2- If (a, b) ϵ R then draw an edge from a to b

3- Remove all Reflexive and Transitive edges

4- Remove the direction of edges and arrange them in the increasing order of heights.

Q Consider a Partial order relation and convert it into hasse diagram?

R = {(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (4,4), (4,8), (8,8)}

Q Consider a Partial order relation and convert it into hasse diagram?

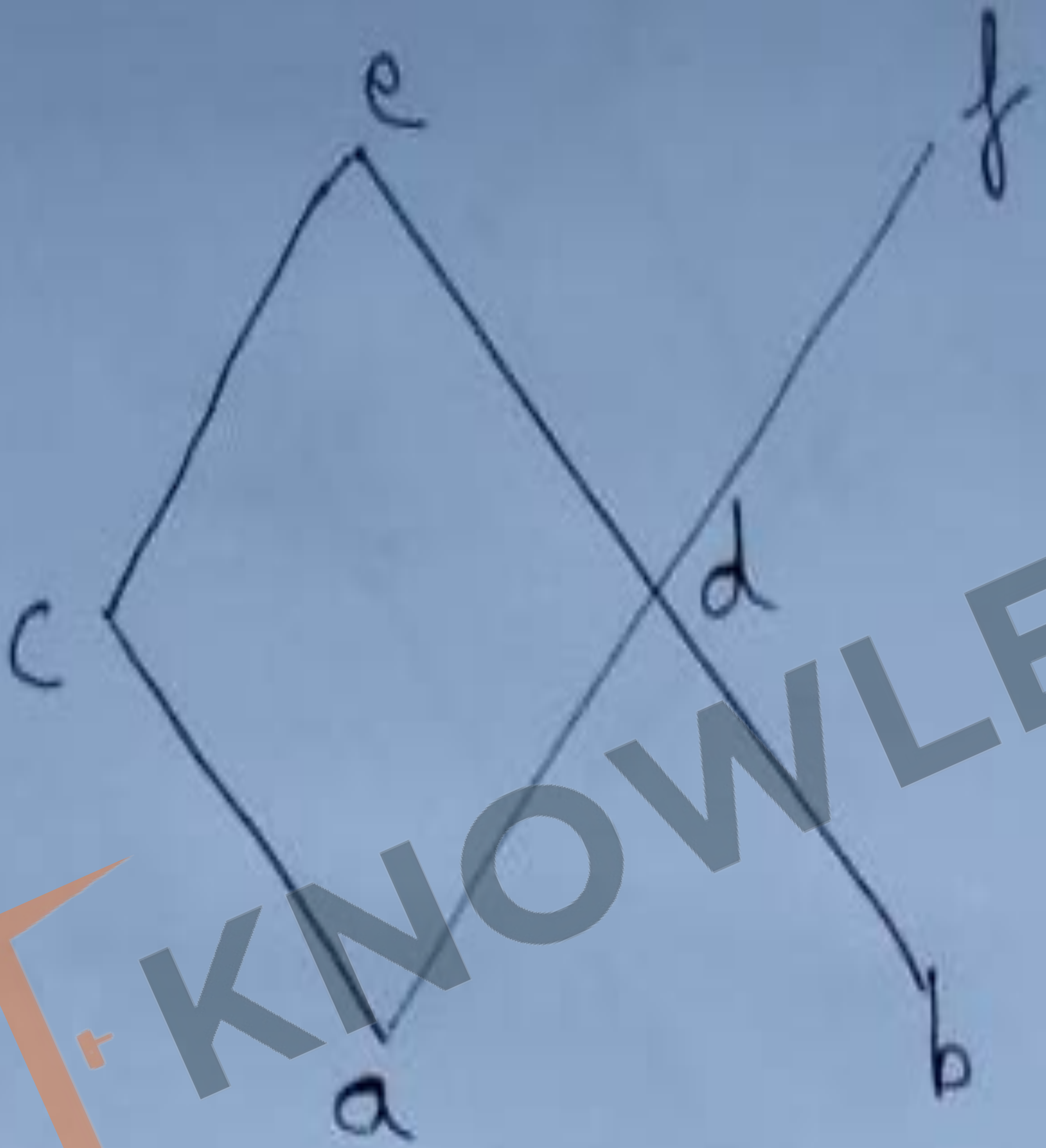
R = {(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)}

Q Let A = (1, 2, 3, 4, 6, 8, 9, 12, 18, 24) be ordered set with relation "x divides y". Draw its Hasse diagram?

Q Draw the Hasse diagram of (A, ≤), where A = (3, 4, 12, 24, 48, 72) and relation ≤ be such that a ≤ b if a divides b.

Q Let A = (2, 3, 6, 12, 24, 36) and relation '≤' be such that x≤y if x divides y. Draw the Hasse diagram of (A, ≤)?

Lattice :- A hasse diagram/Partial order relation is called Lattice if their exist a Join and Meet for every pair of element. Or A hasse diagram/Partial order relation is called Lattice if it is both Join Semi Lattice and Meet Semi Lattice.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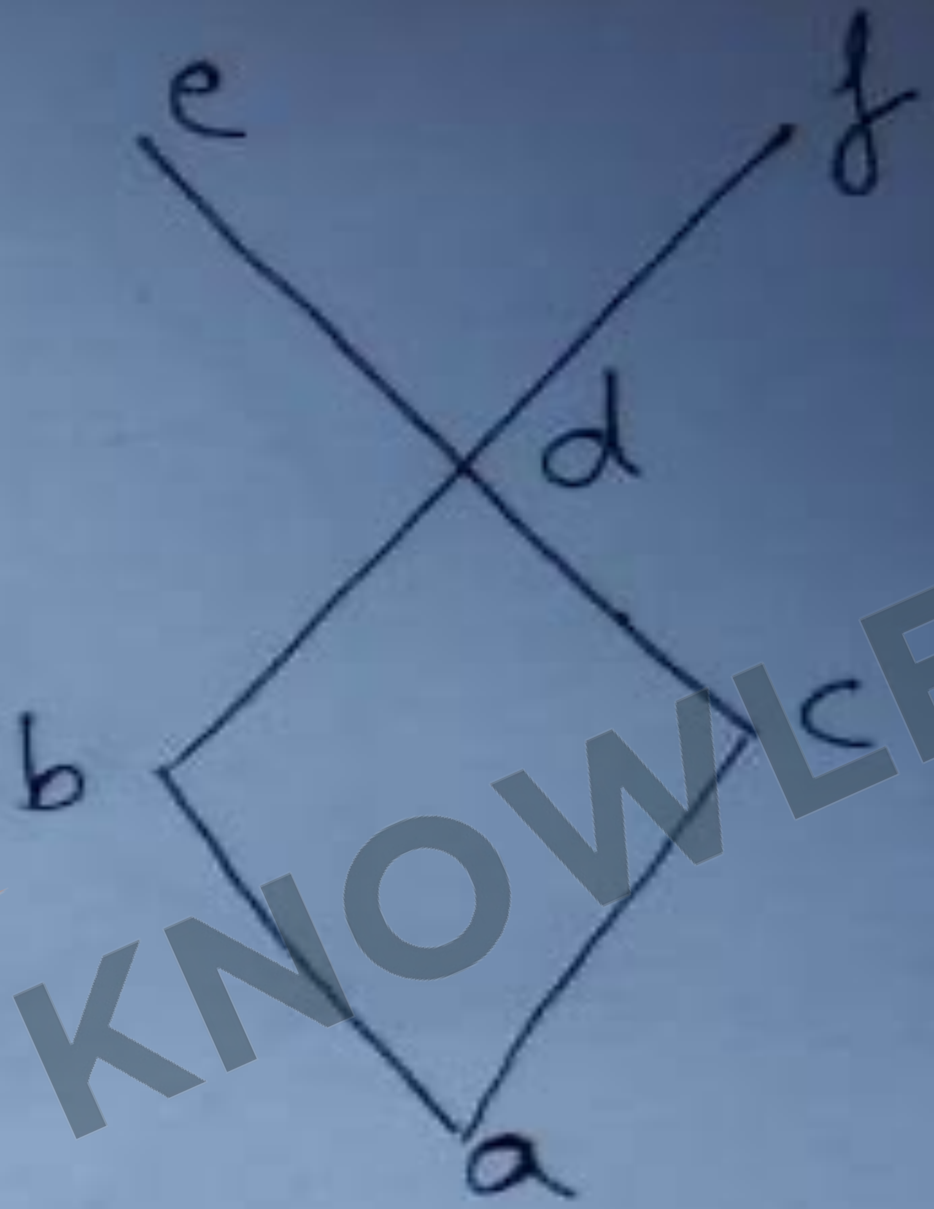
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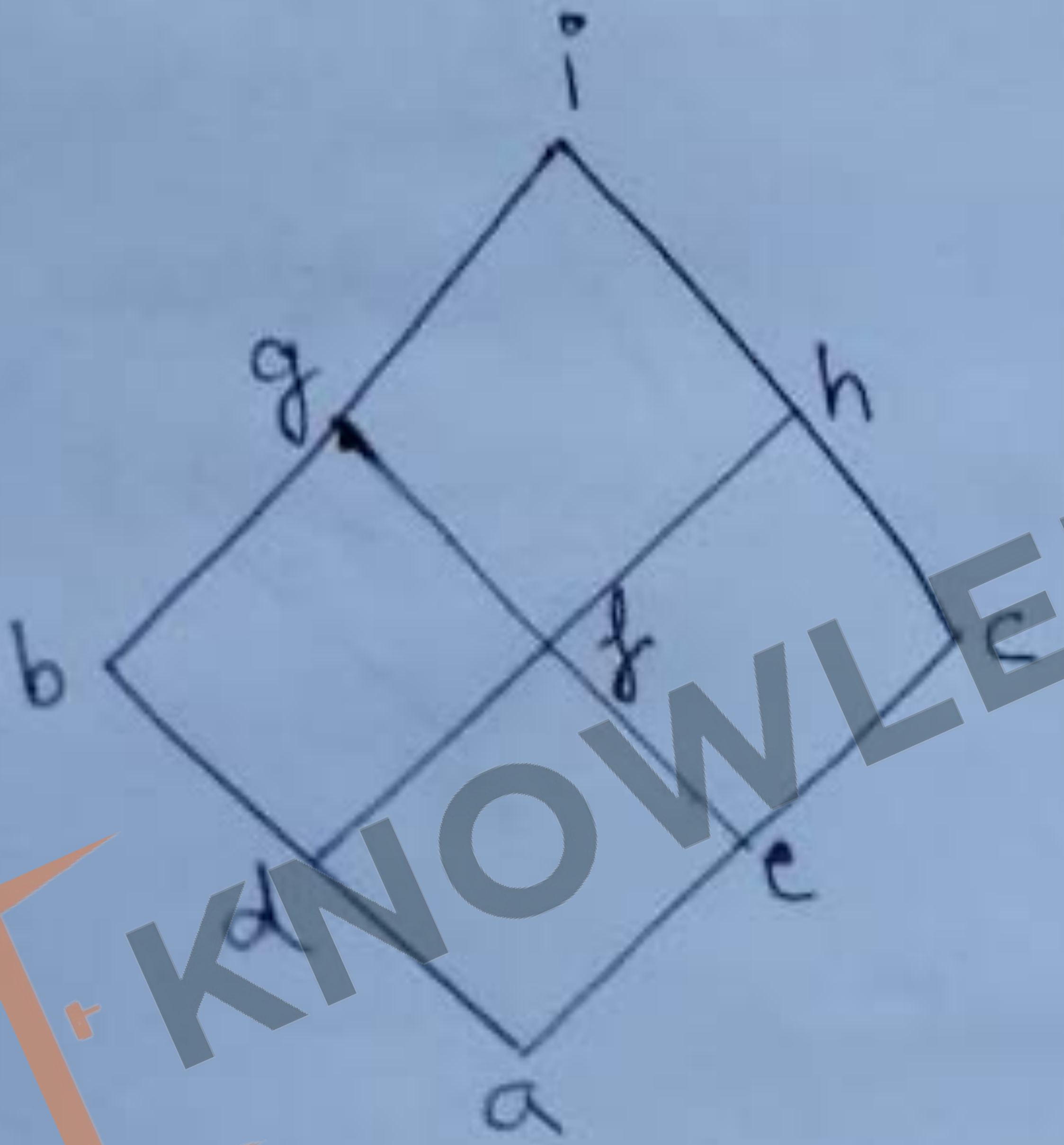

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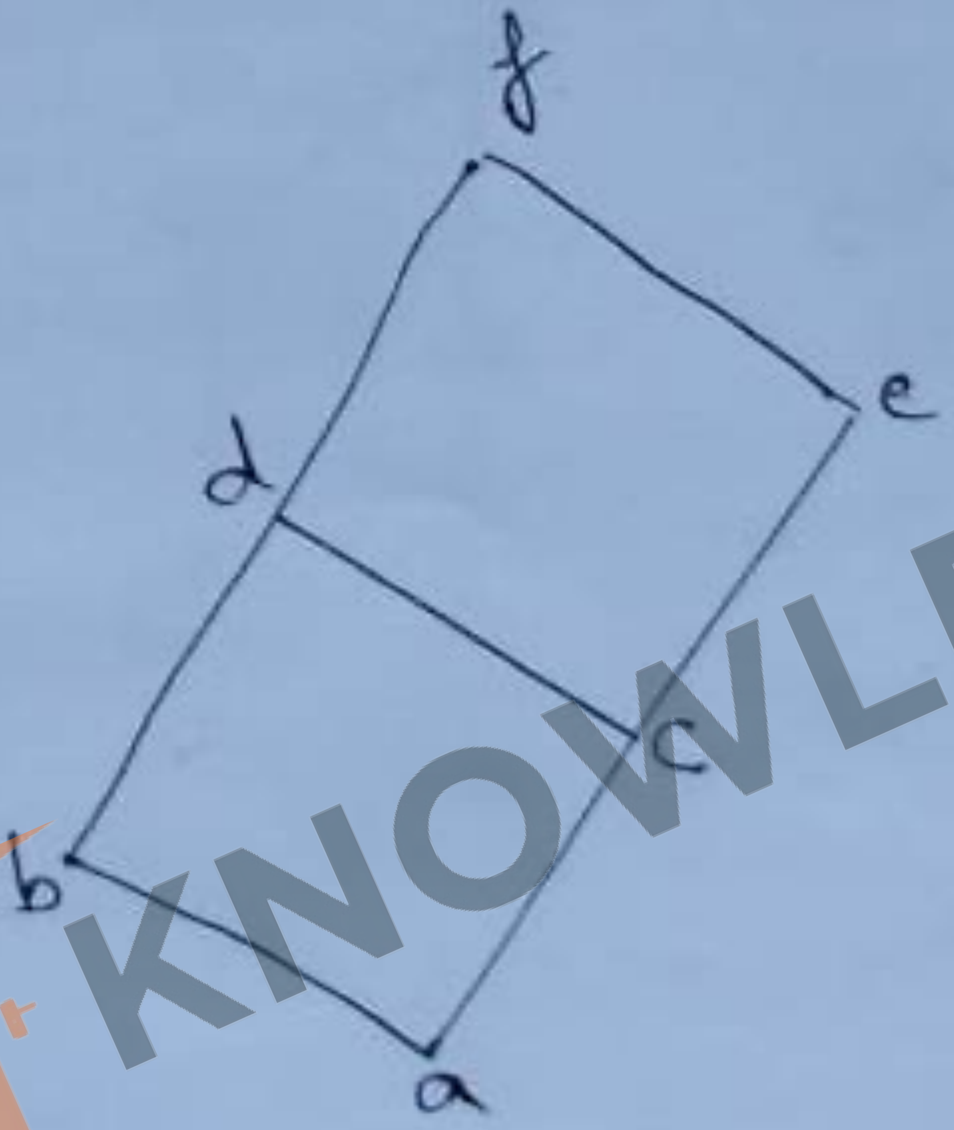
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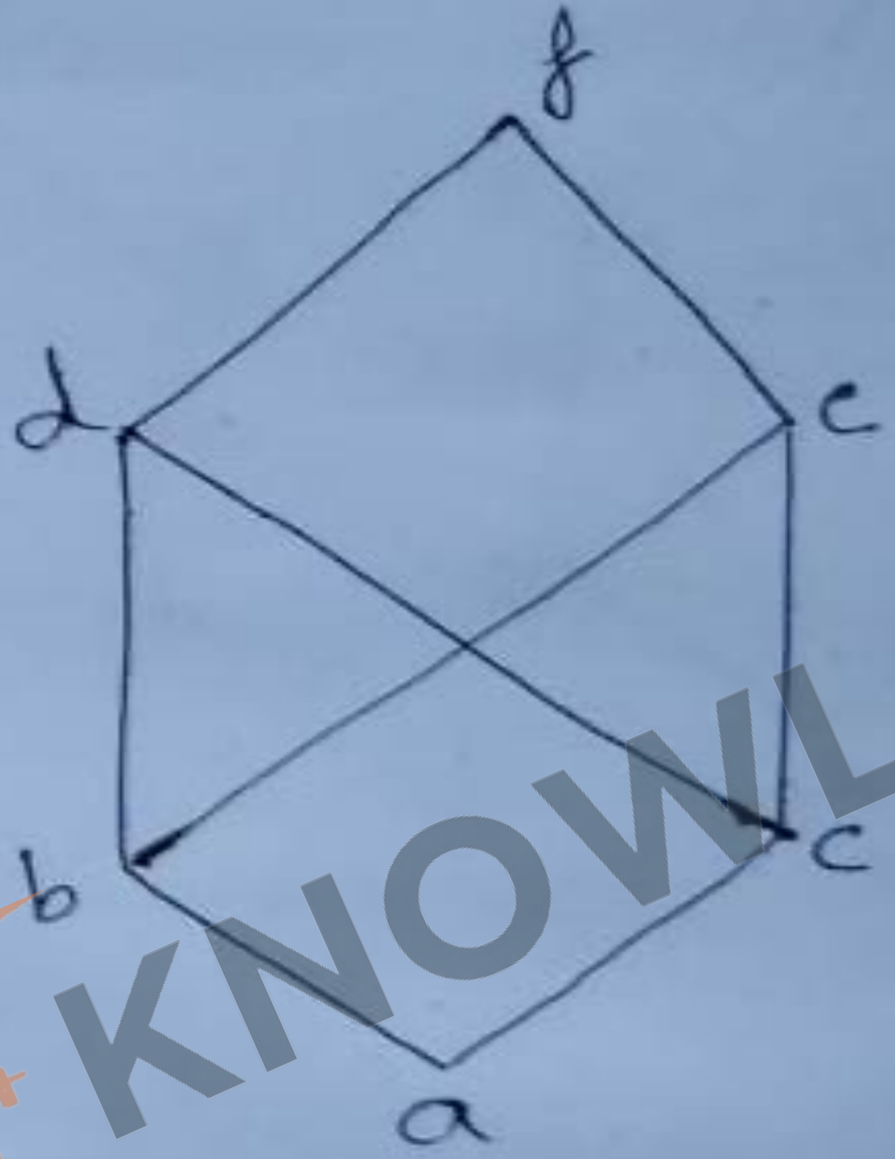
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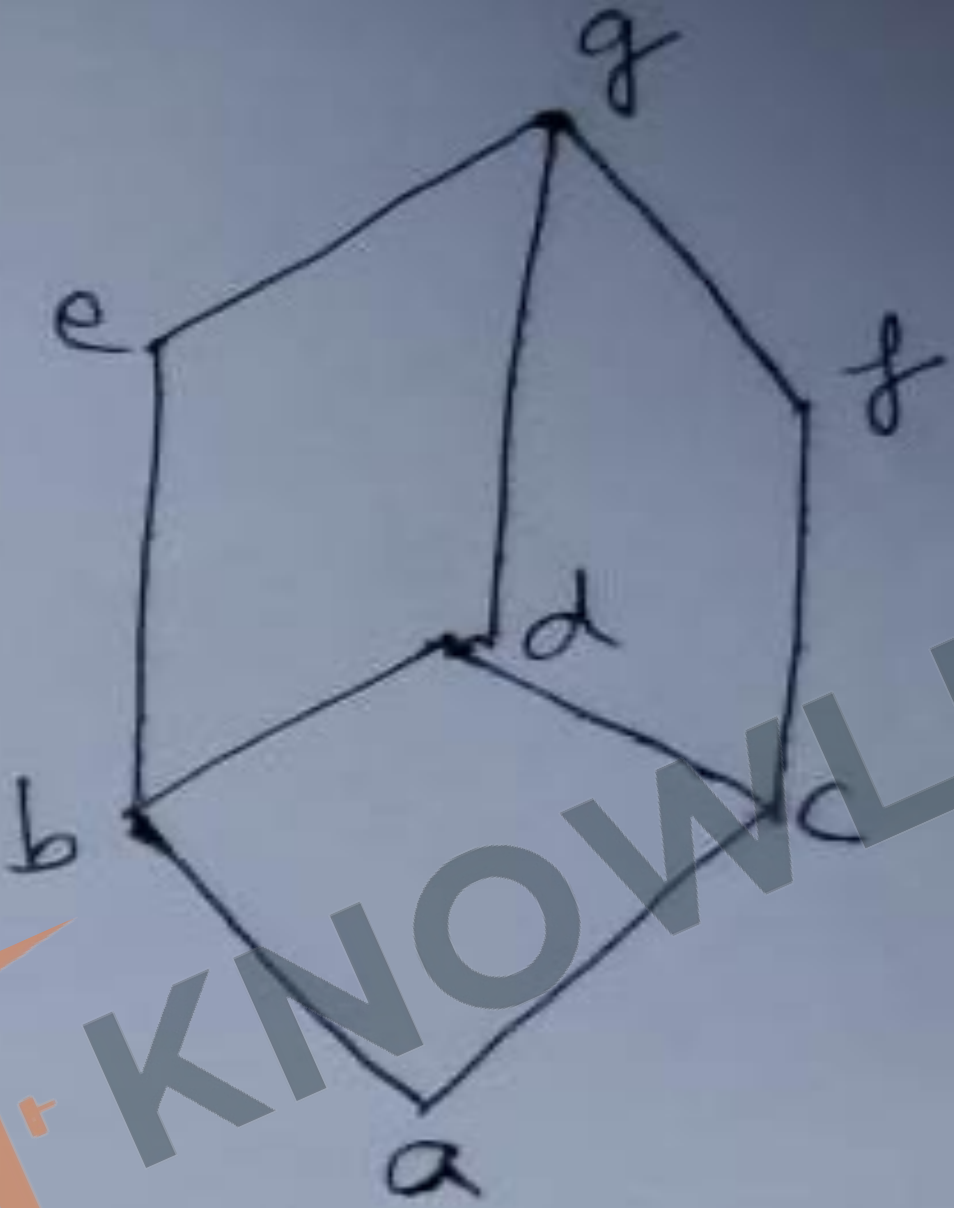
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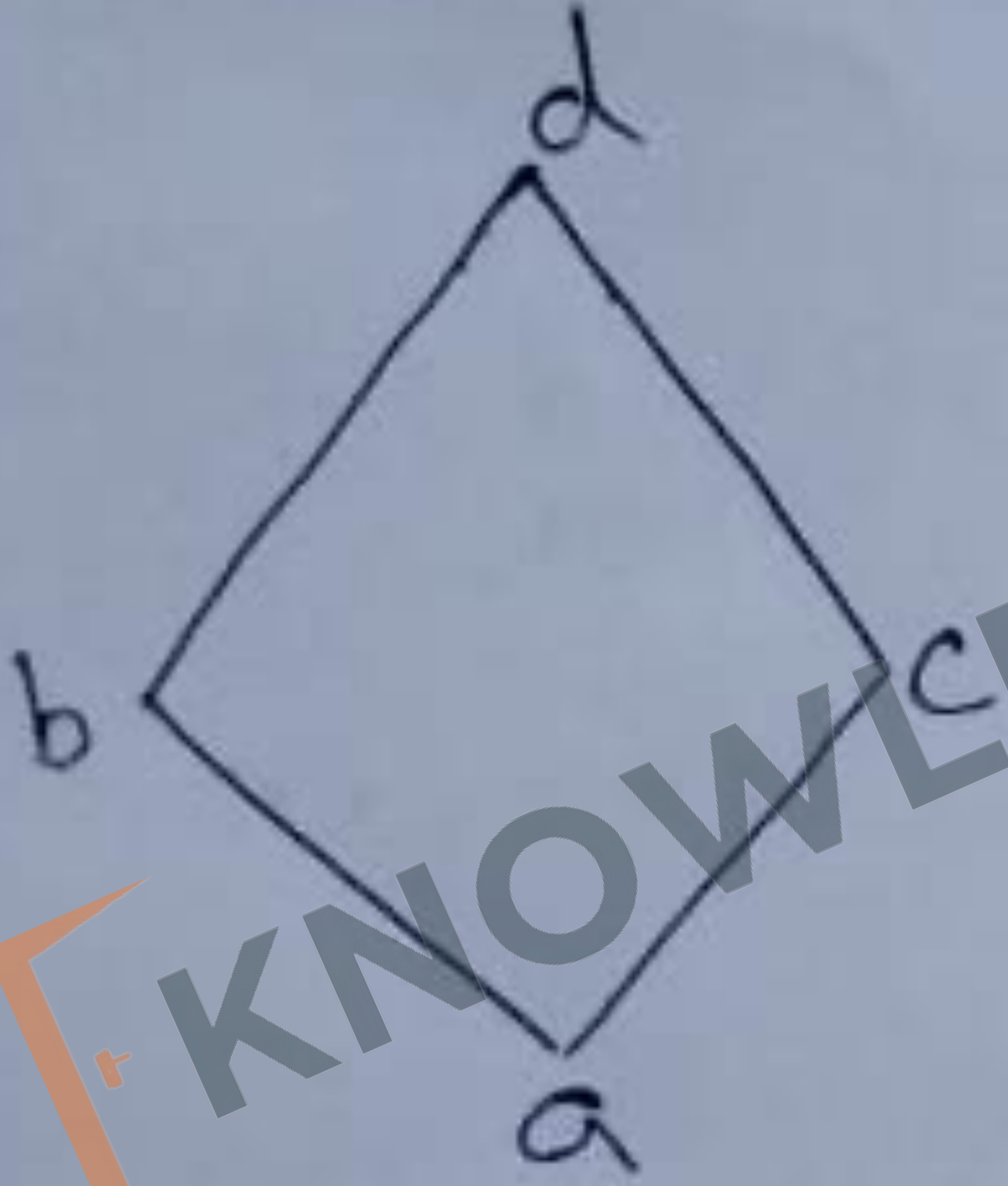
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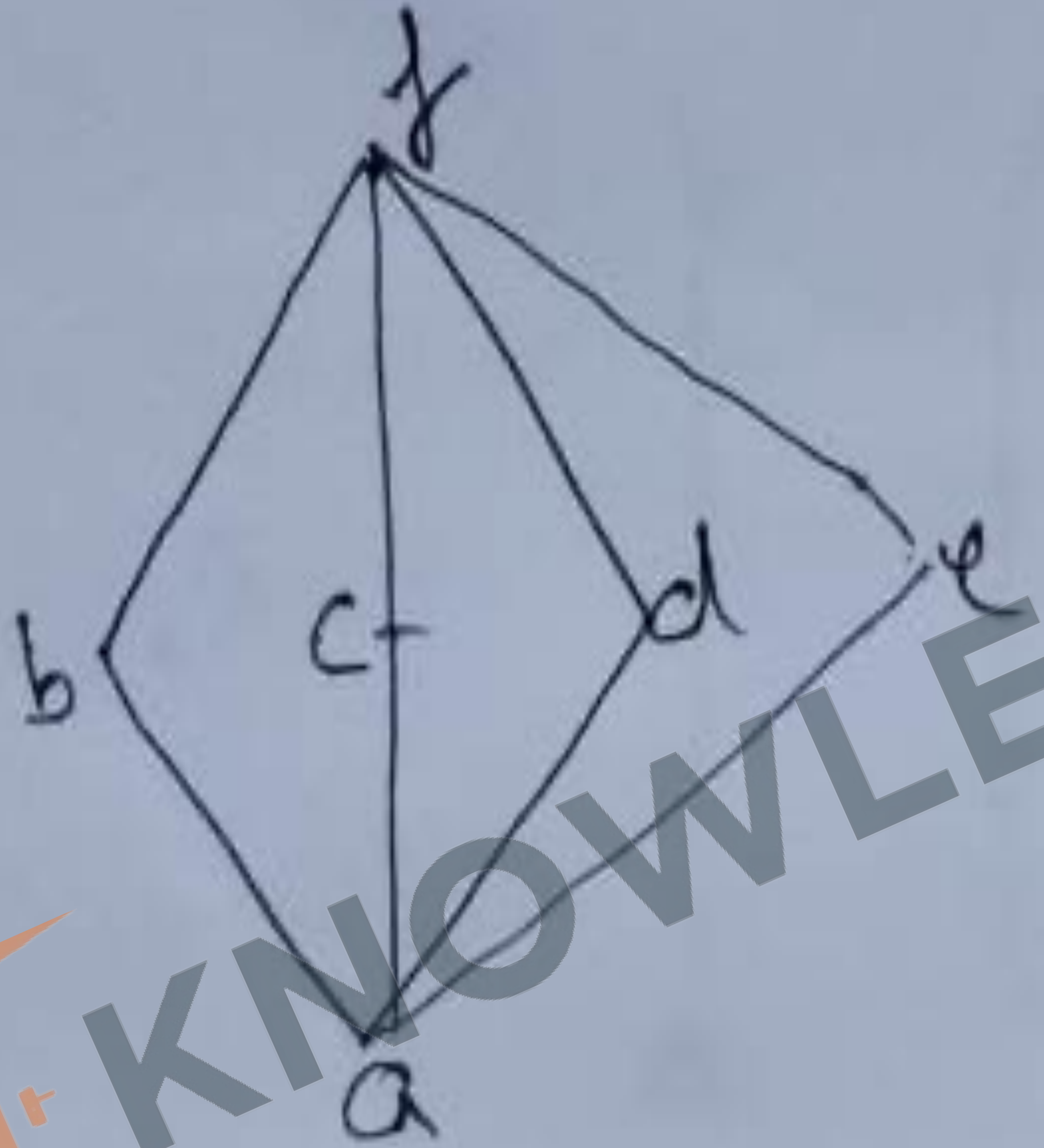
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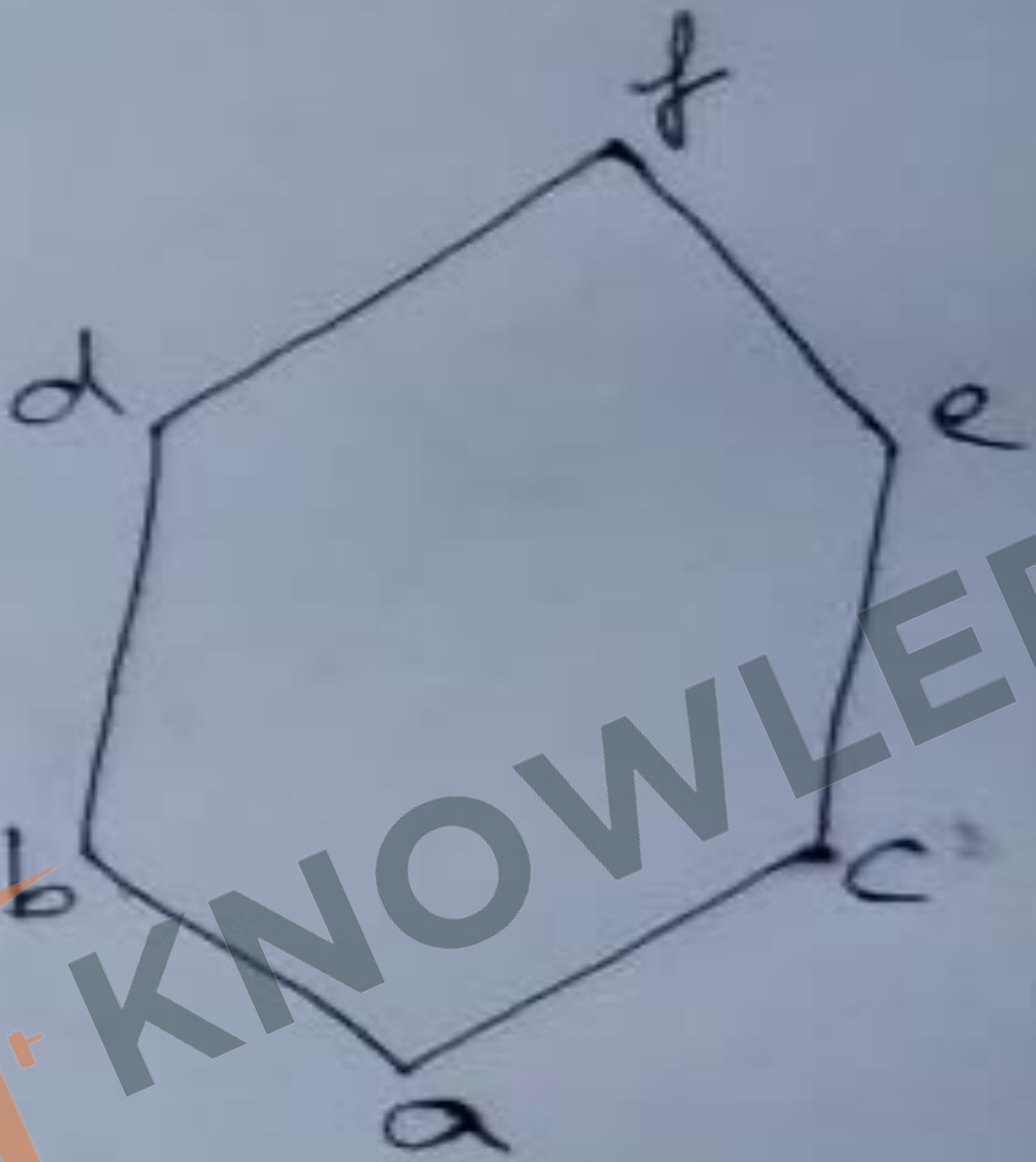
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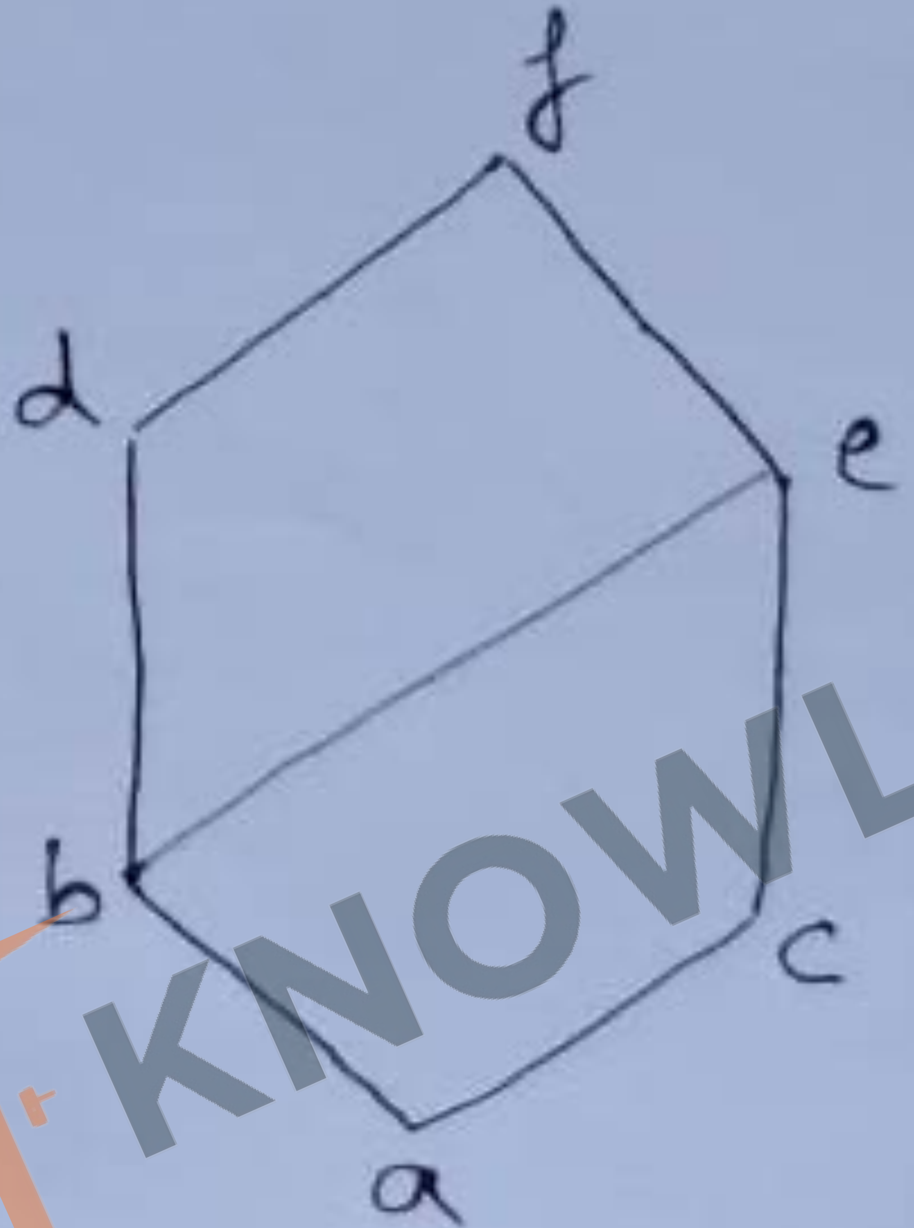
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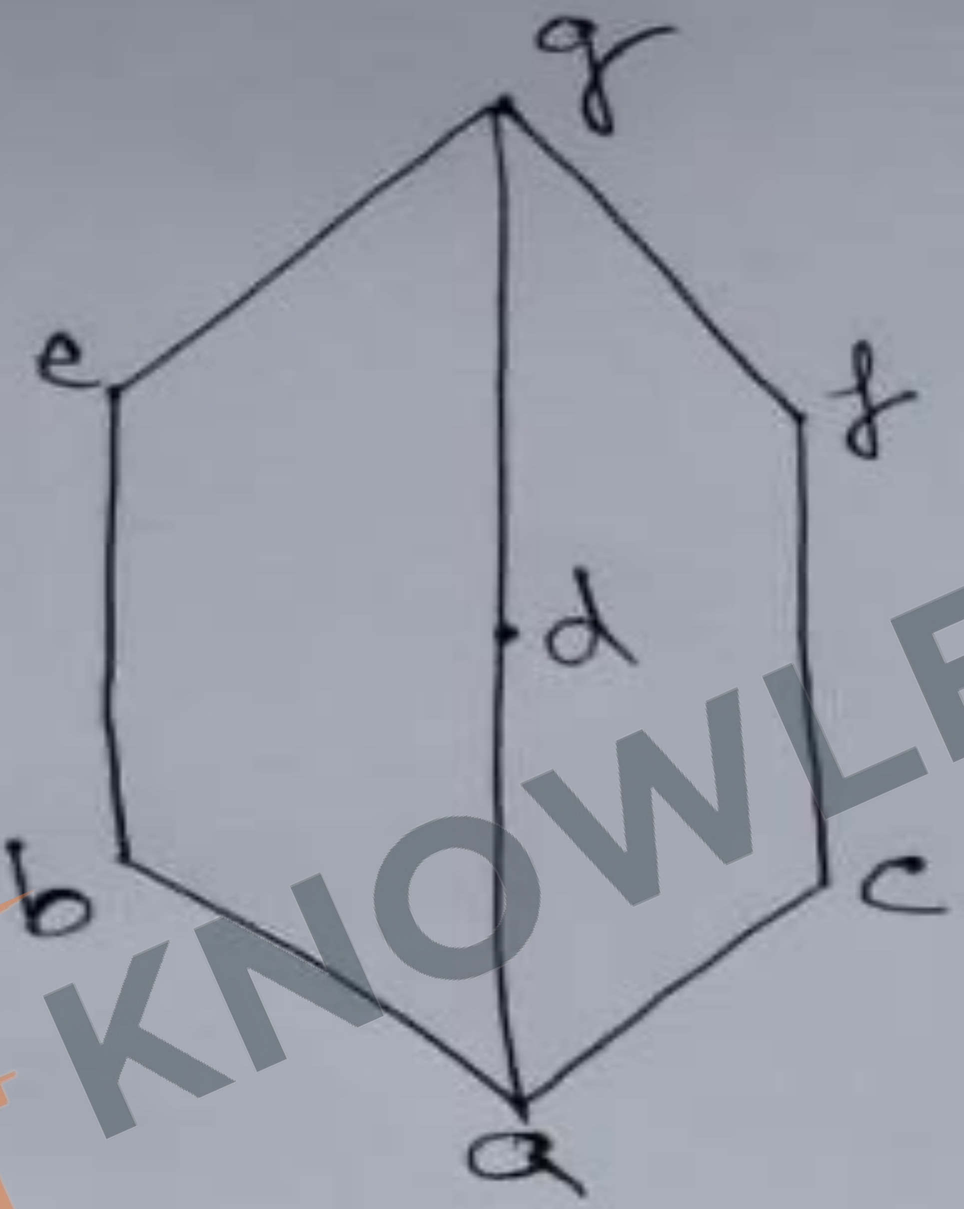
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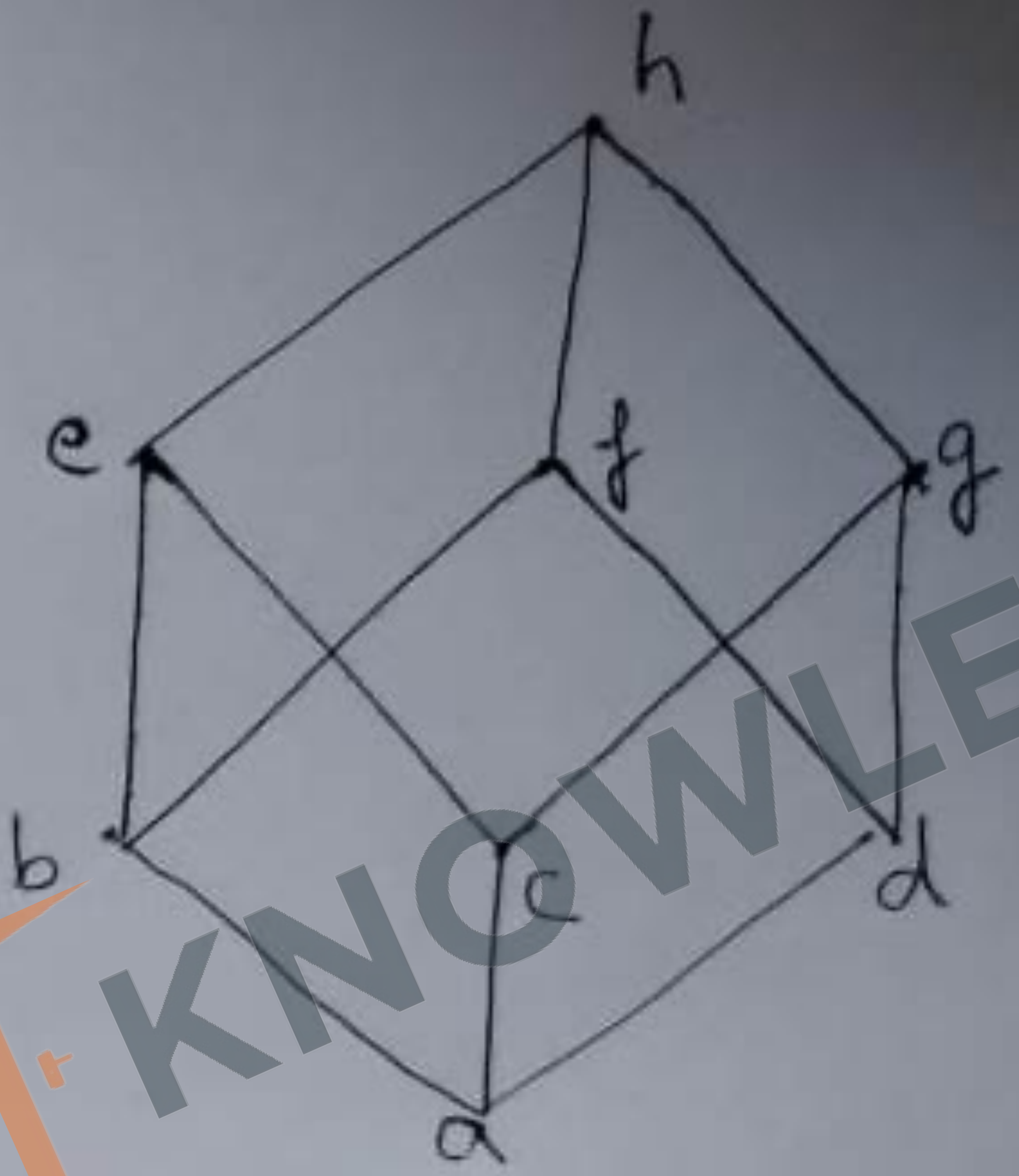
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Idempotent law

- $A \vee A = A$
- $A \wedge A = A$

Associative law

- $(A \vee B) \vee C = A \vee (B \vee C)$
- $(A \wedge B) \wedge C = A \wedge (B \wedge C)$

Commutative law

- $A \vee B = B \vee A$
- $A \wedge B = B \wedge A$

Distributive law

- $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
- $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

De Morgan's law

- $(A \vee B)^C = A^C \wedge B^C$
- $(A \wedge B)^C = A^C \vee B^C$

Identity law

- $A \vee \phi = A$
- $A \wedge \phi = \phi$
- $A \vee U = U$
- $A \wedge U = A$

Complement law

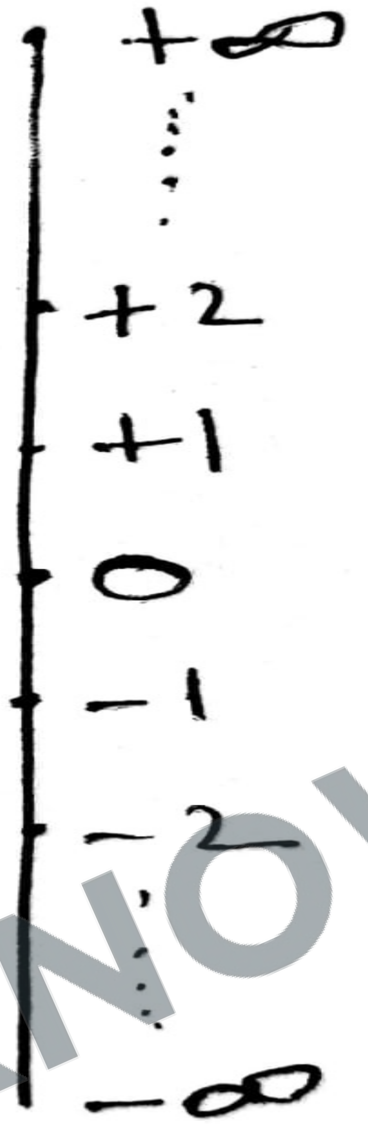
- $A \vee A^C = U$
- $A \wedge A^C = \phi$
- $U^C = \phi$
- $\phi^C = U$

Involution law

- $((A)^C)^C = A$

Boolean algebra

- **Unbounded Lattice :-** If a lattice has infinite of elements then it is called Unbounded Lattice.



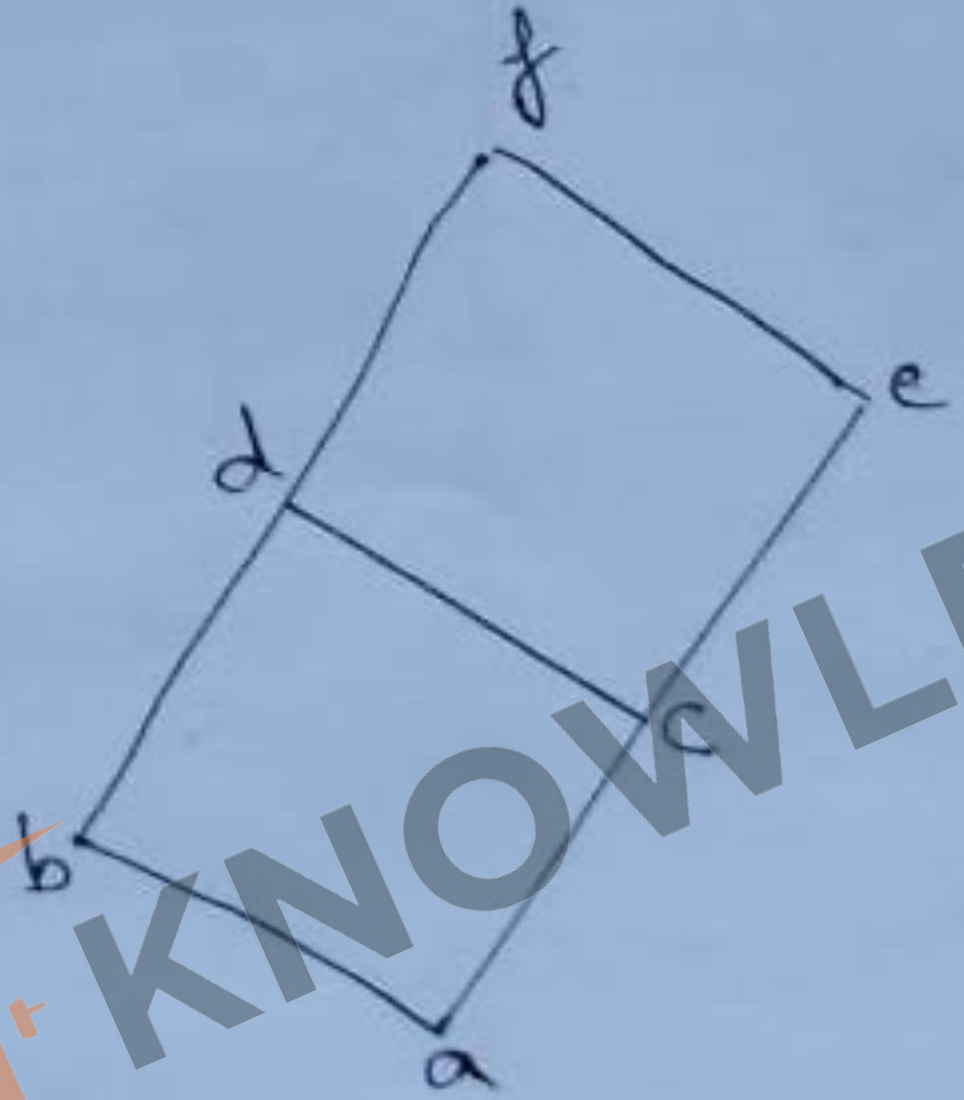
- **Bounded Lattice :-** If a lattice has finite number of elements then it is called Bounded lattice, there will be upper and lower bound in lattice.

- **Complement of an element in a Lattice :-** If two elements a and a^c, are complement of each other, then the following equations must always holds good.
  $a \vee a^c$ = Upper bound of lattice
  $a \wedge a^c$ = Lower bound of lattice

- **Distributive Lattice :-** A lattice is said to be distributed lattice. if for every element their exist at most one completemt(zero or one).

- **Complement Lattice :-** A Lattice is said to be Complement lattice. if for every element their exist at least one complement(one or more).

- **Boolean Algebra :-** A Lattice is said to be Boolean Algebra, if for every element their exist exactly one complement. Or if a lattice is both both complemented and distributed then it is called Boolean Algebra.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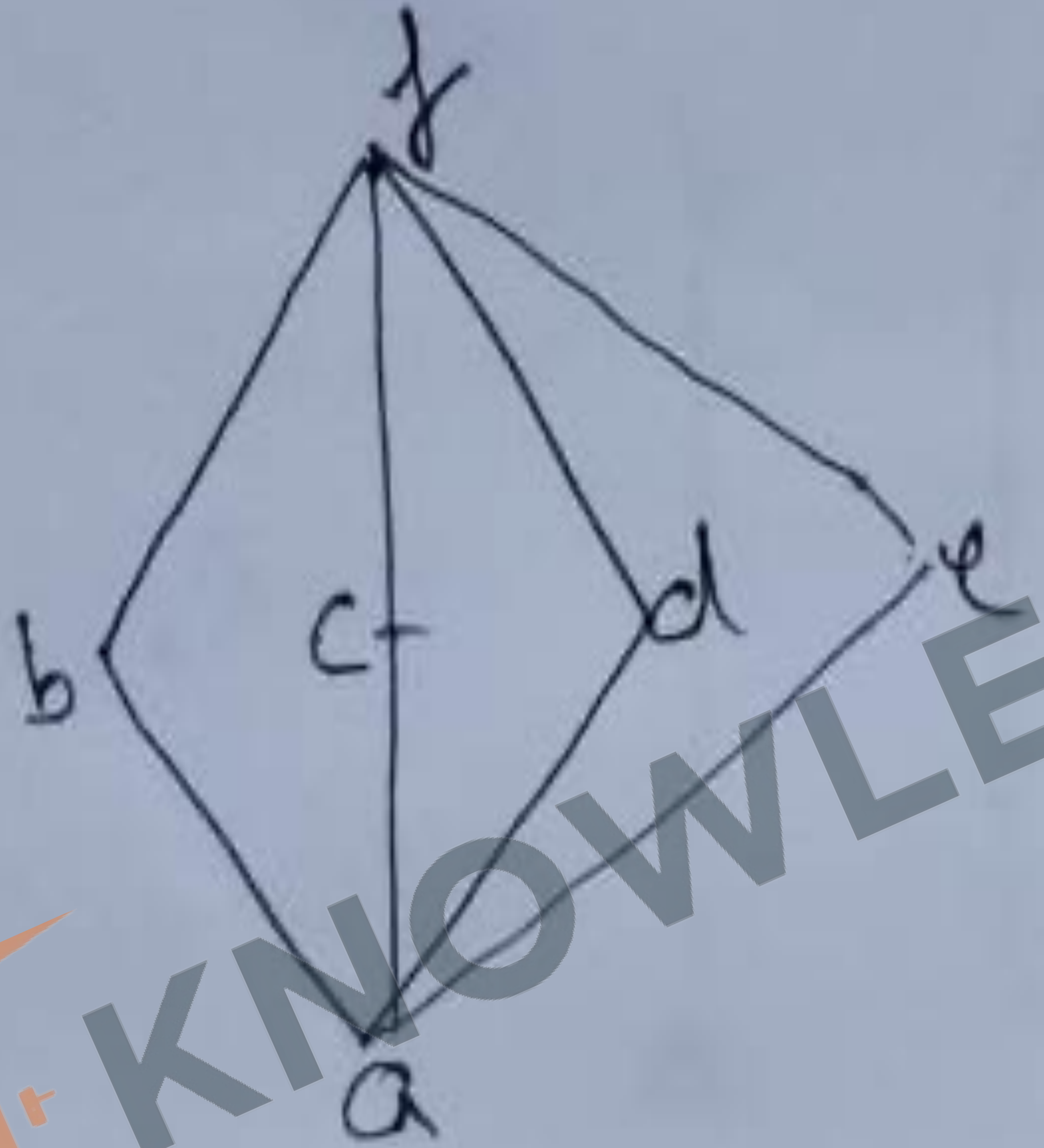
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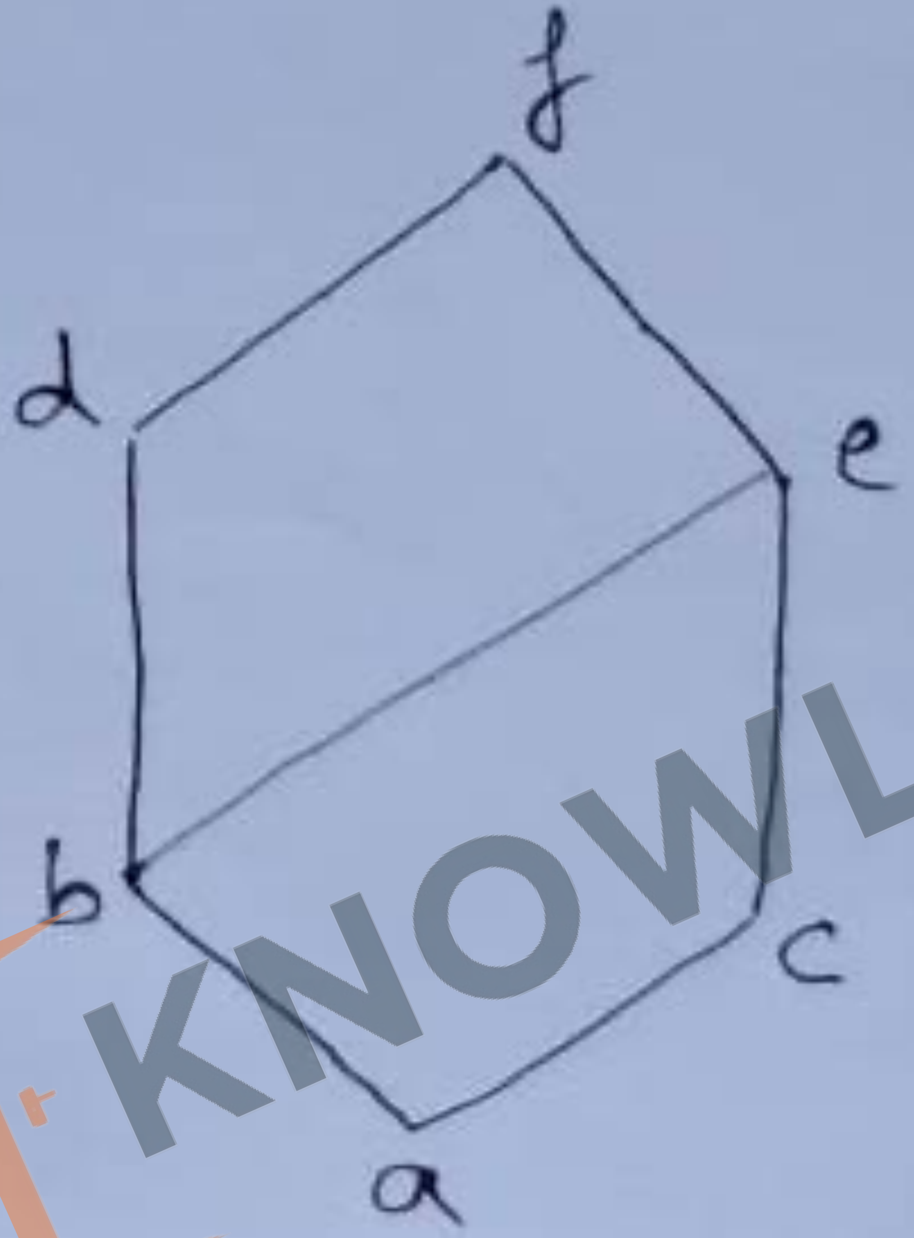

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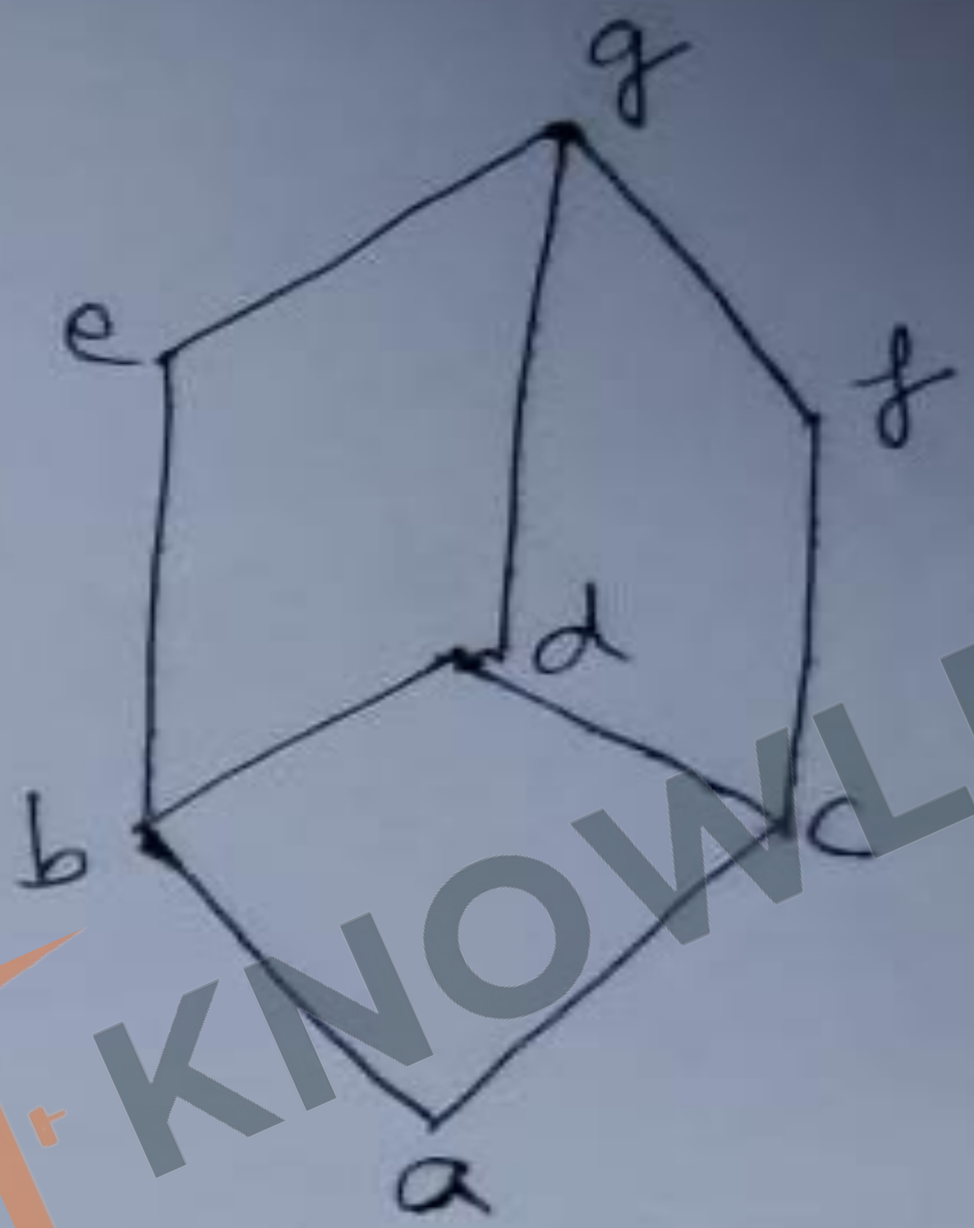
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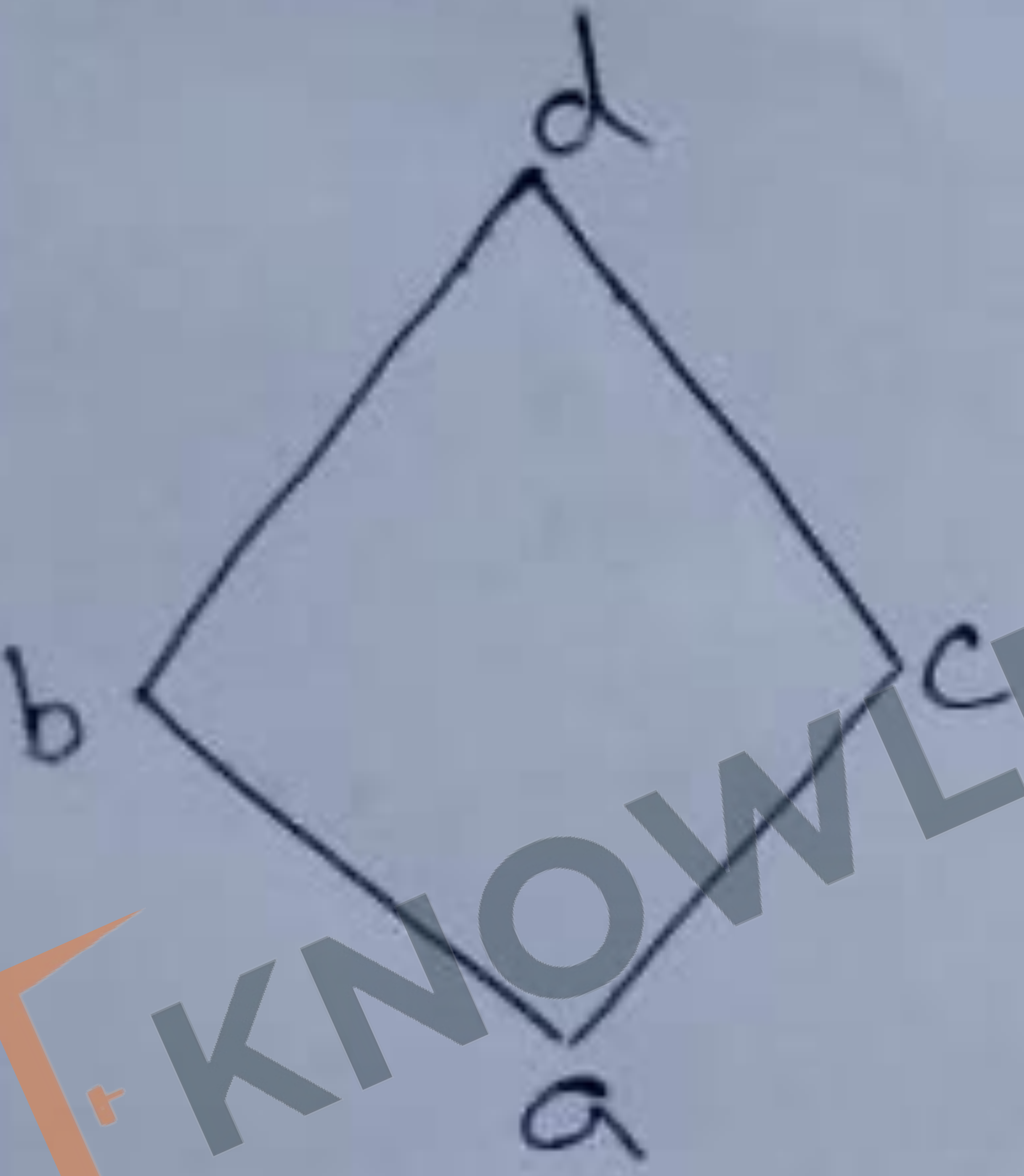
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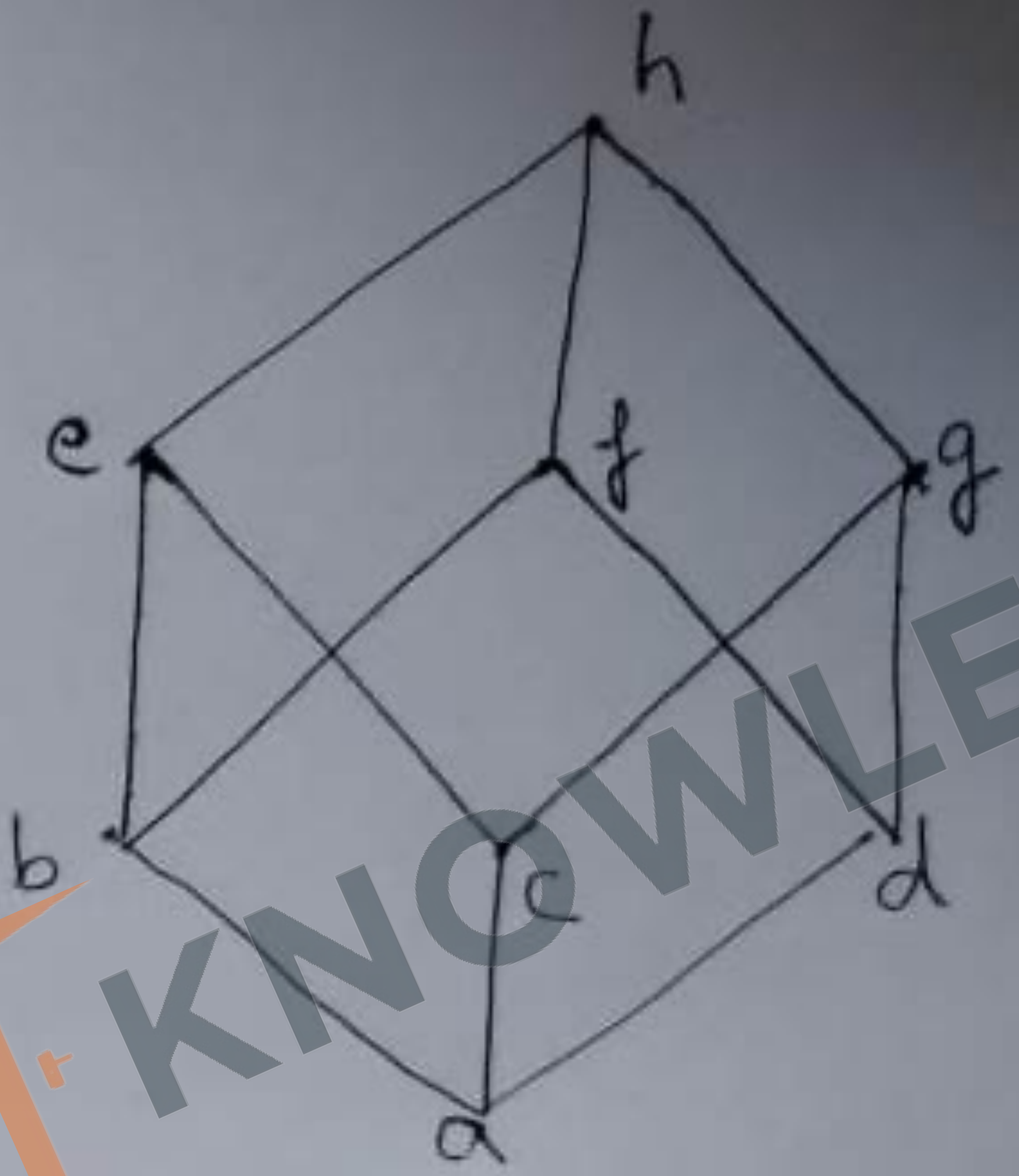
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Q Find which of the following is a lattice and Boolean Algebra?

(1) $[D_{10}, /]$

(2) $[D_{12}, /]$

(3) $[D_{30}, /]$

(4) $[D_{45}, /]$

(5) $[D_{64}, /]$

(6) $[D_{81}, /]$

(7) $[D_{91}, /]$

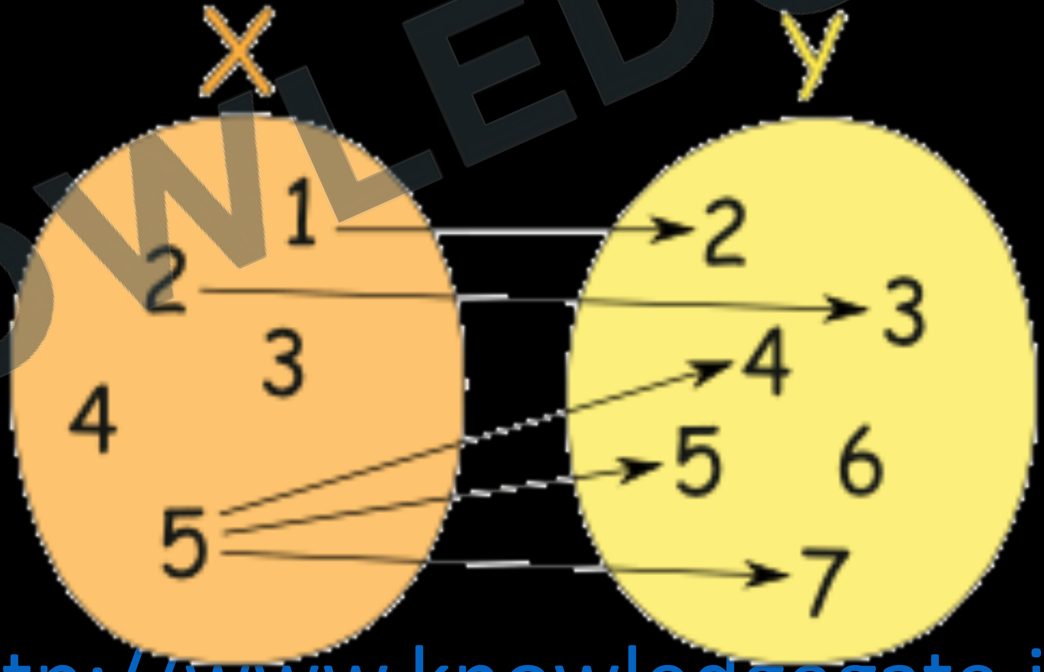
(8) $[D_{110}, /]$

Chapter-4 (Functions)

Function

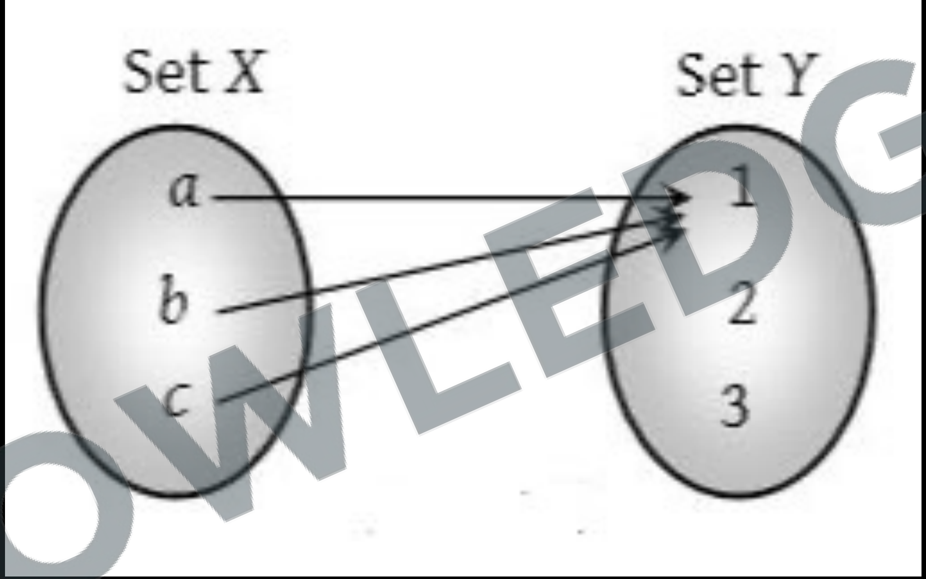
- Functions are widely used in science, and in most fields of mathematics. It has been said that functions are "the central objects of investigation" in most fields of mathematics.

- Let X = {1, 2, 3, 4, 5} and Y = {2, 3, 4, 5, 6, 7} and $R \sqsubseteq X * Y$. Now this is a valid relation but not a function, because there is a element which is not participating in the relationship secondly 5 is relating with more than one element.



Function

- In mathematics, a function is a relation between sets that associates to every element of a first set exactly one element of the second set.
- A relation ‘f’ from a set ‘A’ to a Set ‘B’ is called a function, if each element of A is mapped with a unique element on B.
- f: A→B

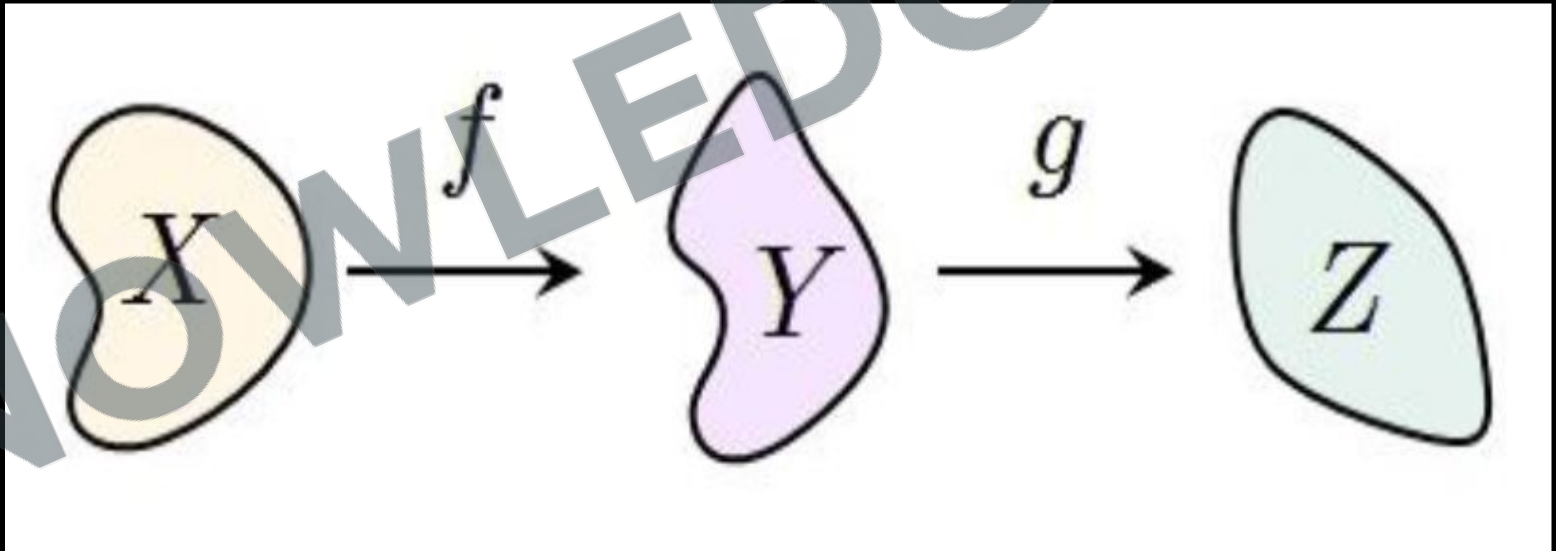


- Range of fun ⊑ B
- Range of f = { y | y ϵ B and (x, y) ϵ f}

- If $|A| = m$ and $|B| = n$, then number of functions possible from A to B = n^m

Function composition

- In mathematics, **function composition** is an operation that takes two functions f and g and produces a function h such that $h(x) = g(f(x))$.
- In this operation, the function g is applied to the result of applying the function f to x. That is, the functions $f: X \to Y$ and $g: Y \to Z$ are **composed** to yield a function that maps x in X to $g(f(x))$ in Z.



- $fog(x) = f(g(x))$
- $gof(x) = g(f(x))$

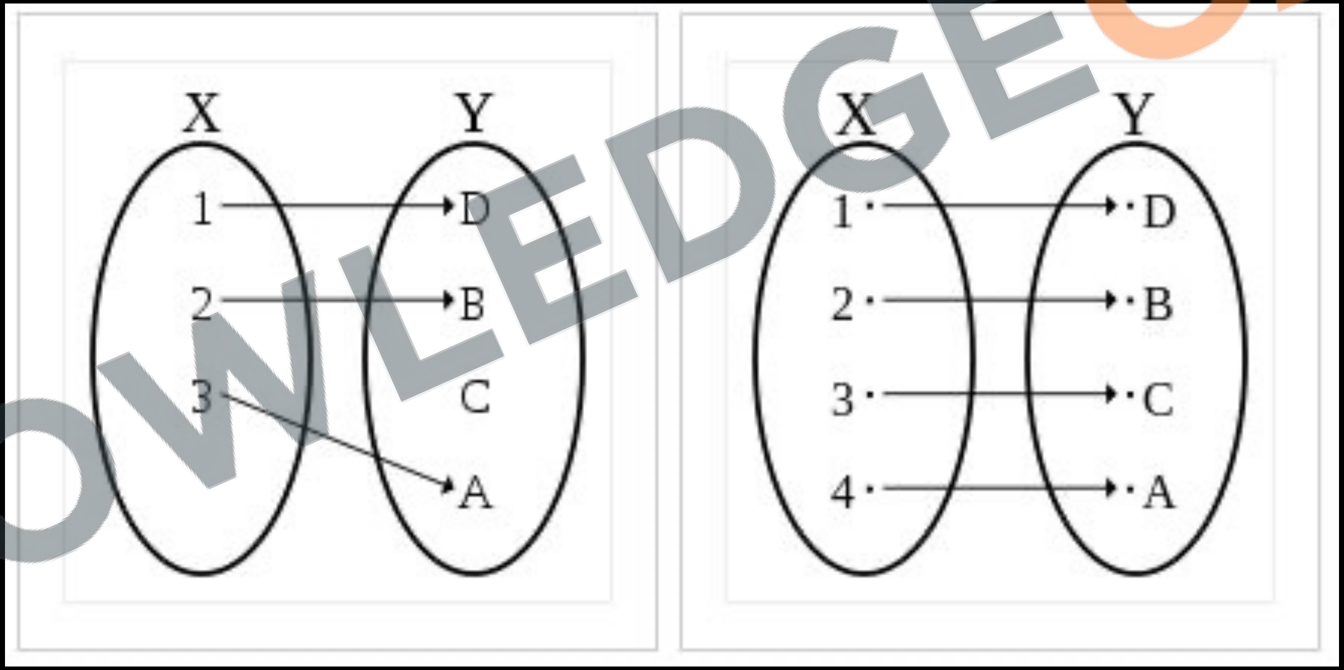
- Composition of functions on a finite set: If f = {(1, 3), (2, 1), (3, 4), (4, 6)}, and g = {(1, 5), (2, 3), (3, 4), (4, 1), (5, 3), (6, 2)}, then $g \circ f$ = {(1, 4), (2, 5), (3, 1), (4, 2)}.
- The composition of functions is always associative—a property inherited from the composition of relations. That is, if f, g, and h are three functions with suitably chosen domains and codomains, then $f \circ (g \circ h) = (f \circ g) \circ h$

One-to-One (Injective Function)

- An **injective function** (also known as **injection**, or **one-to-one function**) is a function that maps distinct elements of its domain to distinct elements of its codomain. In other words, every element of the function's codomain is the image of *at most* one element of its domain.

One-to-One (Injective Function)

- A function F: A→B is said to be one-to-one function if every element of A has distinct image in B
- If A and B are finite set, then one-to-one from A→B is possible
  - if |A| <= |B|



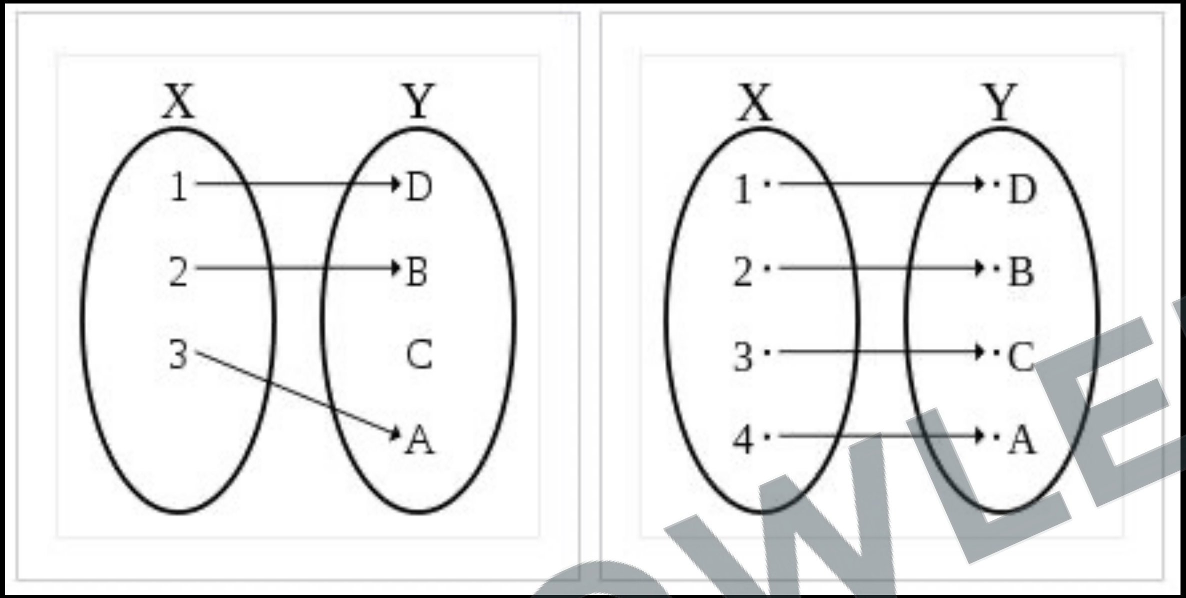
- No of function possible = ${}^{n}p_{m}$ = P (n, m)
- If |A| = |B| = n, then no of functions possible is n!

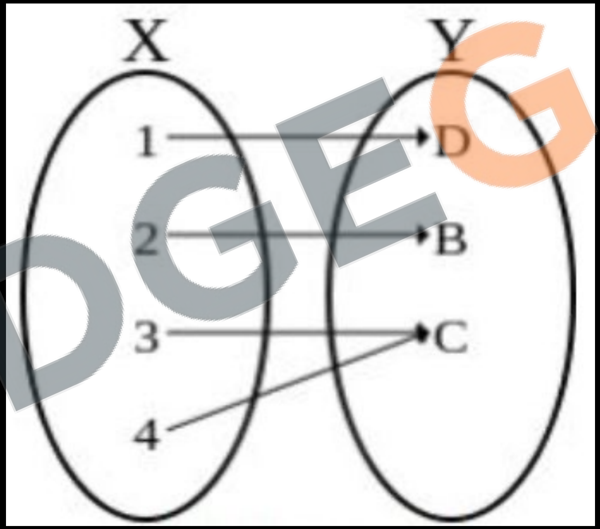
Onto (Surjective Function)

- A function f from a set X to a set Y is **surjective** (also known as **onto**, or a **surjection**), if for every element y in the co-domain Y of f, there is at least one element x in the domain X of f such that $f(x) = y$. It is not required that x be unique; the function f may map one or more elements of X to the same element of Y.

Onto (Surjective Function)

- A function f: A → B is said to be onto if and only if every element of B is mapped by at least one element of A.
- Range of f = B





- If A and B are finite sets, then onto function from A→B is possible, |B|<=|A|
- If |A| = |B|, then every onto function from A to B is also one-to-one function.

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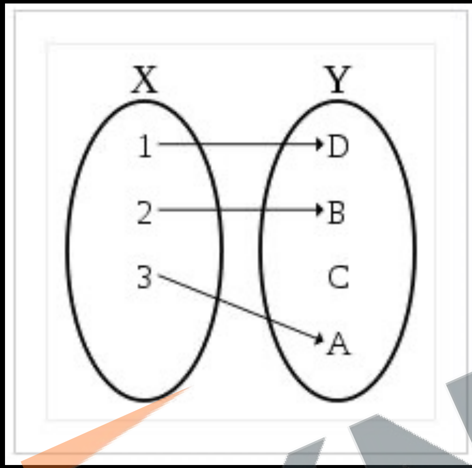
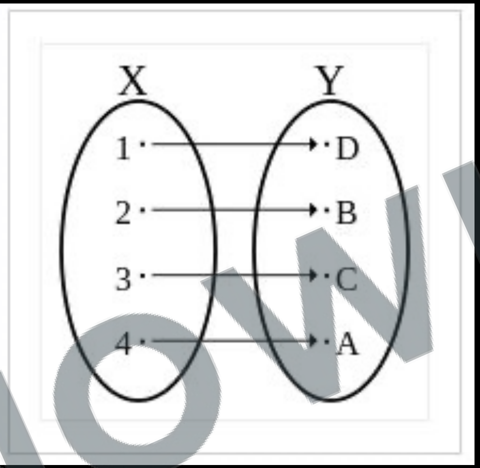
No of onto function possible from A to B

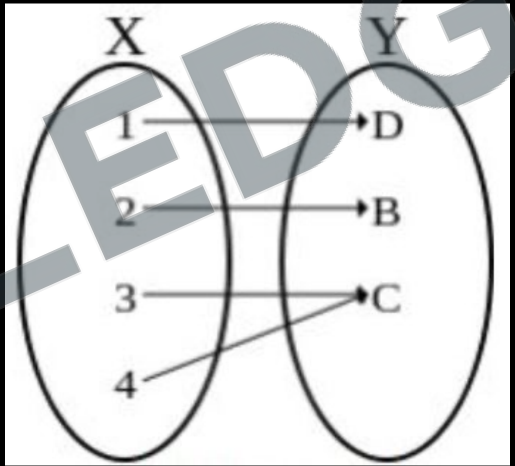
$$= n^m - {}^nc_1(n-1)^m + {}^nc_2(n-2)^m - {}^nc_3(n-3)^m + \text{---------} + (-1)^n\ {}^nc_{n-1}\ 1^m$$

Bijective Function

- In mathematics, a **bijection, bijective function, one-to-one correspondence**, or **invertible function**, is a function between the elements of two sets, where each element of one set is paired with exactly one element of the other set, and each element of the other set is paired with exactly one element of the first set. There are no unpaired elements.





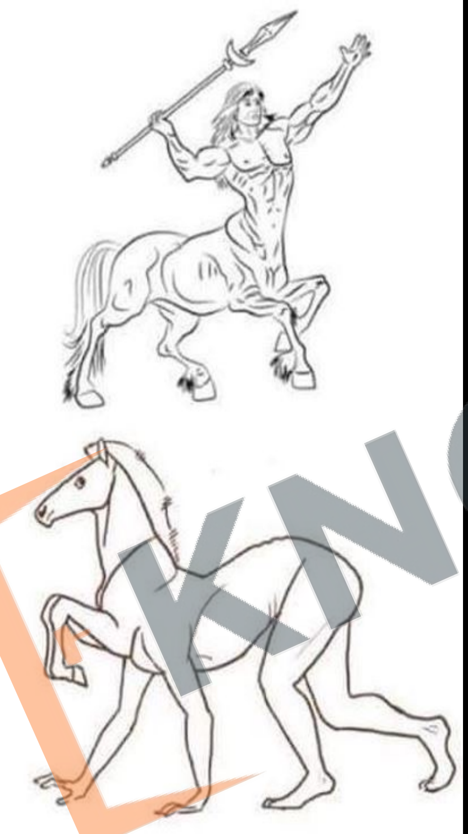


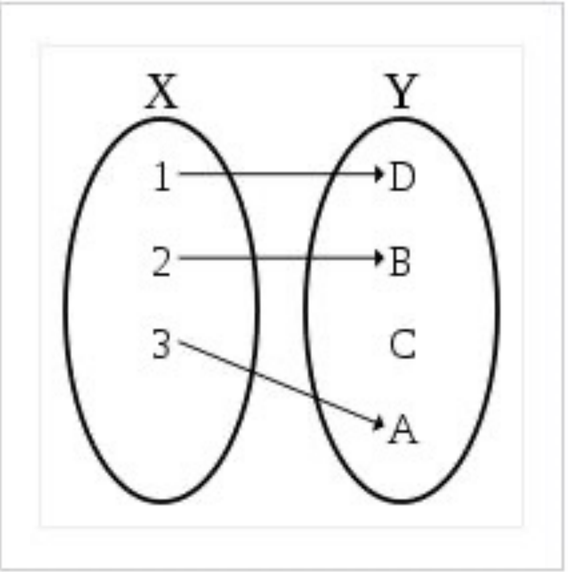
- A function f: A→B is said to be bijection if f is one-to-one and onto.
- Bijection from A and B is possible, if |A|□|B|
- No of Bijection from A to B = ?

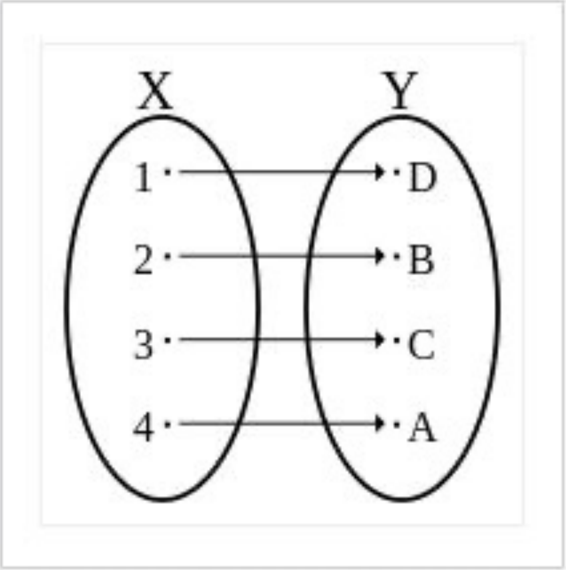
- A function f: A→B is said to be bijection if f is one-to-one and onto.
- Bijection from A and B is possible, if $|A| = |B|$
- No of Bijection from A to B = n!

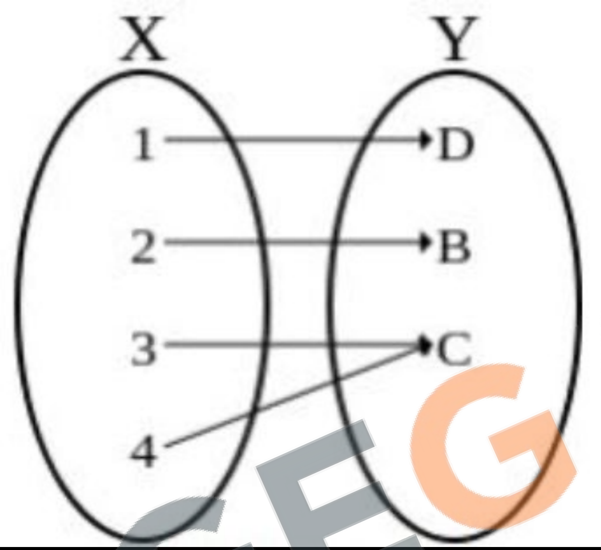
Inverse of a function

- In mathematics, an **inverse function** (or **anti-function**) is a function that "reverses" another function
- If the function f applied to an input x gives a result of y, then applying its inverse function f^{-1} to y gives the result x, and vice versa.
- $f(x) = y$ then $f^{-1}(y) = x$.









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- $f(x) = 5x - 7$
- $f^{-1}(y) = (y + 7)/5$

Q Let *R* denote the set of real numbers. Let $f: R \times R \rightarrow R \times R$ be a bijective function defined by $f(x, y) = (x + y, x - y)$. The inverse function of f is given by

a) $f^{-1}(x, y)=(1 / (x + y), 1 / (x - y))$

b) $f^{-1}(x, y)=(x - y, x + y)$

c) $f^{-1}(x, y)=((x + y) / 2, (x - y) / 2)$

d) $f^{-1}(x, y)=[2(x - y),2(x + y)]$

Chapter-5 (Theory of Logics)

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- A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.
    1. Delhi is the capital of USA
    2. How are you doing
    3. 5<= 11
    4. Temperature is less than 10 C
    5. It is cold today
    6. Read this carefully
    7. X + y = z

- Premises(proposition) is always considered to be true. Premises is a statement that provides reason or support for the conclusion(proposition).
- If a set of Premises(P) yield another proposition Q(Conclusion), then it is called an Argument.
- An argument is said to be valid if the conclusion Q can be derived from the premises by applying the rules of inference.

$\{P_1, P_2, P_3 \ldots, P_N\} \mid- Q$	P_1 P_2 P_3 . . . P_N …… Q …….	$\{P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_N\} \mid- Q$

Types of proposition

1. We use letters to denote **propositional variables** (or **statement variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables.

2. The conventional letters used for propositional variables are *p, q, r, s.* The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

3. Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

Operators / Connectives

1. **Negation: -** let p be a proposition, then negation of p new proposition, denoted by $\neg p$, is the statement "it is not the case that p".

2. The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p. e.g. ¬"Michael's PC runs Linux" = "It is not the case that Michael's PC runs Linux." = "Michael's PC does not run Linux."

Negation	
P	¬P
F	
T	

Conjunction

- Let *p* and *q* be propositions. The *conjunction* of *p* and *q*, denoted by $p \wedge q$, is the proposition "*p* and *q*."

- The conjunction $p \wedge q$ is true when both *p* and *q* are true and is false otherwise.

Conjunction		
p	q	p ∧ q
F	F	
F	T	
T	F	
T	T	

Q consider the following arguments and find which of them are valid?

1	
P_1	$(p \wedge q)$
Q	p

Simplification

2	
P_1	P
Q	$p \wedge q$

3	
P_1	$\neg(p \wedge q)$
P_2	P
Q	$\neg q$

Conjunctive Syllogism

4	
P_1	$\neg(p \wedge q)$
P_2	q
Q	$\neg p$

Disjunction

- Let *p* and *q* be propositions. The *disjunction* of *p* and *q*, denoted by $p \lor q$, is the proposition. "*p* or *q*." The disjunction $p \lor q$ is false when both *p* and *q* are false and is true otherwise.

Disjunction		
p	q	p ∨ q
F	F	
F	T	
T	F	
T	T	

Q consider the following arguments and find which of them are valid?

1	
P_1	$(p \wedge q)$
Q	$p \vee q$

2	
P_1	$p \vee q$
Q	$(p \wedge q)$

5	
P_1	$(p \vee q)$
P_2	$\neg p$
Q	q

6	
P_1	$(p \vee q)$
P_2	$\neg q$
Q	p

Disjunctive syllogism

✓ Addition

$$\frac{p}{p \vee q}$$

Implication

1. Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q". The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

2. In conditional statement $p \rightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion.

3. The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds.

Implication		
p	q	p q
F	F	
F	T	
T	F	
T	T	

p	q	$P \rightarrow q$	$\neg p$	$\neg q$	$\neg P \rightarrow \neg q$	$q \rightarrow p$	$\neg q \rightarrow \neg p$	$\neg p \vee q$
F	F							
F	T							
T	F							
T	T							

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1. $p \rightarrow q$ *implication*

2. $q \rightarrow p$ *converse*

3. $\neg p \rightarrow \neg q$ *inverse*

4. $\neg q \rightarrow \neg p$ *contra positive*

1. $p \to q = \neg q \to \neg p$

2. $p \to q$ will be true if either p is false or q is true, $p \to q = \neg p \vee q$

Q Express Converse, Inverse and Contrapositive of the following Statement "If $x + 5 = 8$ then $x = 3$"

Q consider the following arguments and find which of them are valid?

Modus Ponens	
P_1	$p \to q$
P_2	P
Q	Q

Modus Tollens	
P_1	$p \to q$
P_2	$\neg Q$
Q	$\neg p$

1	
P_1	$\neg p$
Q	$p \to q$

2	
P_1	q
Q	$p \to q$

Bi-conditional

- Let *p* and *q* be propositions. The bi*conditional statement* $p \leftrightarrow q$ is the proposition.
  - " *p if and only* q".
- The biconditional statement $p \leftrightarrow q$ *is true when p and q have the same values, and false otherwise. Biconditional statements are also called bi-implications.*
  - "*p* is necessary and sufficient for *q*"
  - "if *p* then *q*, and conversely"
  - "*p* iff *q*."
  - $p \leftarrow\rightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

Bi-conditional		
p	q	P q
F	F	
F	T	
T	F	
T	T	

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Q Construct the truth table for the following statement ?

$(P \rightarrow \neg Q) \rightarrow \neg P$

Q The statement $(\neg p) \implies (\neg q)$ is logically equivalent to which of the statement below?

1) $p \implies q$ **2)** $q \implies p$ **3)** $(\neg q) \lor (p)$ **4)** $(\neg p) \lor q$

Type of cases

- **Tautology/valid: -** A propositional function which is always having truth in the last column, is called tautology. E.g. $\boldsymbol{p \vee \neg p}$

p	$\neg p$	$p \vee \neg p$
F	T	
T	F	

- **<u>Contradiction/ Unsatisfiable</u>: -** A propositional function which is always having false in the last column, is called Contradiction. E.g. $\boldsymbol{p \wedge \neg p}$

p	$\neg p$	$p \wedge \neg p$
F	T	
T	F	

- **Contingency:** **-** A propositional function which is neither a tautology nor a contradiction, is called Contingency. E.g. $\boldsymbol{p} \vee \boldsymbol{q}$

p	q	p ∨ q
F	F	
F	T	
T	F	
T	T	

- **Satisfiable: -** A propositional function which is not contradiction is satisfiable. i.e. it must have at least one truth value in the final column e.g. $p \lor q$

- **Functionality Complete Set: -** A set of connectives is said to be functionally complete if it is able to write any propositional function.
  - $\{\wedge, \neg\}$
  - $\{\vee, \neg\}$

Q consider the following argument

I_1: if today is Gandhi ji's birthday, then today is oct 2nd

I_2: today is oct 2nd

C: today is Gandhi ji's birthday

Q consider the following argument

I_1: if Canada is a country, then London is a city

I_2: London is not a city

C: Canada is not a country

Q Consider the following logical inferences.

I_1: If it rains then the cricket match will not be played.
The cricket match was played.
Inference: There was no rain.

I_2: If it rains then the cricket match will not be played.
It did not rain.
Inference: The cricket match was played.

Q Consider the following propositional statements:

P_1 : $((A \wedge B) \rightarrow C)) \equiv ((A \rightarrow C) \wedge (B \rightarrow C))$

Q Consider the following propositional statements:

$P_2 : ((A \vee B) \rightarrow C)) \equiv ((A \rightarrow C) \neg (B \rightarrow C))$

Q Use rules of inference to justify that the three hypotheses
i. "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on."
ii. "If the sailing race is held, then the trophy will be awarded."
iii. "The trophy was not awarded." imply the conclusion
iv. "It rained."

Show that $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \vee \mathbf{r})$ is tautology without using truth table.

Q "If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect". Test the validity of this argument using truth table.

Q Prove the validity of the following argument.

If Mary runs for office, she will be elected. If Mary attends the meeting, she will run for office. Either Mary will attend the meeting or she will go to India. But Mary cannot go to India."Thus Mary will be elected".

Q Prove the validity of the following argument.
If Mary runs for office, she will be elected. If Mary attends the meeting, she will run for office. Either Mary will attend the meeting or she will go to India. But Mary cannot go to India."Thus Mary will be elected".

Q Prove the validity of the following argument "if the races are fixed so the casinos are cooked, then the tourist trade will decline. If the tourist trade decreases, then the police will be happy. The police force is never happy. Therefore, the races are not fixed."

Q Prove the validity of the following argument "If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job, or I will not work hard."

- **Canonical SOP form**: - In a sum of product form expression, if each AND term (product term) consists all the literals(variables) appearing either in complements or uncomplemented form. E.g. a’bc + ab’c’ + abc. Then the form is said to be canonical SOP. Here we have Principle Disjunction Normal Form

- **Canonical POS form**: - In a product of sum form expression, if each OR term (sum term) consists all the literals(variables) appearing either in complements or uncomplemented form. E.g. (a’ + b + c). (a + b’ + c’). (a + b + c). Then the form is said to be Canonical POS form. Here we have Principle Conjunctive Normal Form.

First order Predicate Logic

- Sometime propositional logic cannot derive any meaningful information even though, we as human can understand that argument is meaningful or not.

- P_1: Every Indian like cricket
- P_2: Sunny is an Indian
- Q: Sunny Likes cricket

- The reason propositional logic fails here because using only inference system we can not conclude Q from P_1 and P_2.

- In first order logic we understand, a new approach of subject and predicate to extract more information from a statement
  - 1 is a natural number (1 is subject, natural number is predicate)
  - we can write FOPL (short hand notation) for this as NatNo(1) = 1 is natural number
  - Similarly, we can understand the meaning of NatNo(2) as 2 is a natural number
  - NatNo(x): x is a natural number

- Sometime subject is not a single element but representing the entire group.
  - Every Indian like Cricket.
  - We can have a propositional function Cricket(x): x likes Cricket.
  - We can fix domain of discussion or universe of discourse as, x is an Indian.

- If i say four Indian are there I_1, I_2, I_3, I_4
- I_1 likes cricket $\wedge$ I_2 likes cricket $\wedge$ I_3 likes cricket $\wedge$ I_4 likes cricket
- $Cricket(I_1) \wedge Cricket(I_2) \wedge Cricket(I_3) \wedge Cricket(I_4)$
- But problem with this notation is as there is 130+ corers Indian this formula will become very long and in some case we actually do not know how many subjects are there in the universe of discourse. so, we again need a short hand formula.
- $\forall_x$ Cricket(x), if we confine x to be Indian then it means every x like cricket.

- **Universal quantifiers**: - The universal quantification of a propositional function is the proposition that asserts
- P(x) is true for all values of x in the universe of discourse.
- The universe of discourse specifies the possible value of x.
- $\forall_x$ P(x), i.e. for all value of a P(x) is true

- Let try some other statement 'Some Indian like samosa'
  - if i say four Indian are there I_1, I_2, I_3, I_4
  - I_1 like samosa $\vee$ I_2 like samosa $\vee$ I_3 like samosa $\vee$ I_4 like samosa
  - $Samosa(I_1) \vee Samosa(I_2) \vee Samosa(I_3) \vee Samosa(I_4)$
  - $\exists_x$ Samosa(x), if we confine x to be Indian then it means some x likes samosa.

- **<u>Existential quantifiers</u>**: - with existential quantifier of a propositional that is true if and only if P(x) is true for at least one value of x in the universe of discourse.

- There exists an element x is the universe of discourse such that P(x) is true.

- $\exists_x$ P(x), i.e. for at least one value of a P(x) is true

- let’s change the universe of discourse from Indian to human
  - if human is Indian then it likes cricket
  - Indian(x): x is an Indian
  - Cricket(x): x likes Cricket
  - if I_1 is Indian then likes cricket ∧ if I_2 is Indian then likes cricket ∧ if I_3 is Indian then likes cricket ∧ if I_4 is Indian then likes cricket
  - $[Indian(I_1) \rightarrow cricket(I_1)] \wedge [Indian(I_2) \rightarrow cricket(I_2)] \wedge [Indian(I_3) \rightarrow cricket(I_3)] \wedge [Indian(I_4) \rightarrow cricket(I_4)]$
  - $\forall_x [Indian(x) \rightarrow cricket(x)]$

- let's change the universe of discourse from Indian to human
  - if human is Indian then it likes samosa
  - Indian(x): x is an Indian
  - Samosa(x): x likes Samosa
  - if I_1 is Indian then likes samosa $\vee$ if I_2 is Indian then likes samosa $\vee$ if I_3 is Indian then likes samosa $\vee$ if I_4 is Indian then likes samosa
  - $[\text{Indian}(I_1) \wedge \text{samosa}(I_1)] \vee [\text{Indian}(I_2) \wedge \text{samosa}(I_2)] \vee [\text{Indian}(I_3) \wedge \text{samosa}(I_3)] \vee [\text{Indian}(I_4) \wedge \text{samosa}(I_4)]$
  - $\exists_x [\text{Indian}(x) \wedge \text{samosa}(x)]$

- let check validity of a statement "Some Indians like samosa" = $\exists_x$ [Indian(x) $\rightarrow$ samosa(x)], x is human
- let human contains four elements I_1, I_2, I_3, I_4 out of which I_1, I_2 are Indian while I_3, I_4 are not Indian
- Suppose I_1, I_2, I_3 do not likes samosa
  - [Indian(I_1) $\rightarrow$ samosa(I_1)] V [Indian(I_2) $\rightarrow$ samosa(I_2)] V [Indian(I_3) $\rightarrow$ samosa(I_3)]
  - [T $\rightarrow$ F] V [T $\rightarrow$ F] V [F $\rightarrow$ F]
  - [F] V [F] V [T]
  - T
- conclusion $\exists_x$ is not used with $\rightarrow$

Negation

- $\neg [\forall_x P(x)] = \exists_x \neg P(x)$
- $\neg [\exists_x P(x)] = \forall_x \neg P(x)$

Universal specification: By this rule if the premise $\forall_x P(x)$ is true then $P(c)$ is true where c is particular member of Universe of Discourse.

$$\forall_x P(x)$$

$$P(c)$$

Universal generalization: By this rule if P(c) is true for all c in Universe of Discourse then $\forall_x P(x)$ is true.

$$P(c)$$

$$\forall_x P(x)$$

Existential specification: By this rule if $\exists_x P(x)$ is true then $P(x)$ is true for some particular member of Universe of Discourse.

$$\exists_x P(x)$$

$$P(c)$$

Existential generalization: By this rule if P(c) is true for some particular member c in Universe of Discourse, then $\exists_x P(x)$ is true

$$P(c)$$

$$\exists_x P(x)$$

Universal modus ponens: By this rule if $P(x) \rightarrow Q(c)$ is true for every x and $P(c)$ is true for some particular member c in Universe of Discourse then $Q(c)$ is true.

$$\forall_x P(x) \rightarrow Q(x)$$
$$P(c)$$

$$Q(c)$$

Universal modus tollens: By this rule if P(x) → Q(x) is true for every x and ~Q(c) is true for some particular c in Universe of Discourse then ~Q(c) is true.

$$\forall_x P(x) \rightarrow Q(x)$$
$$\sim Q(c)$$

$$\sim P(c)$$

Q The CORRECT formula for the sentence is

“not all rainy days are cold”

$$\exists_d\,(Rainy(d) \land {\sim}Cold(d))$$

Q Consider the statement

"Not all that glitters is gold"

$$\exists_x: glitters(x) \land \neg gold(x)$$

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Q Consider the statement

"None of my friends are perfect."

$$\neg\exists_x(F(x) \land P(x))$$

Q Consider the statement

“Some real numbers are rational”

$$\exists_x (\text{real}(x) \land \text{rational}(x))$$

Q Consider the statement

“Gold and silver ornaments are precious”.

$$\forall_x ((G(x) \lor S(x)) \rightarrow P(x))$$

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Q Consider the statement

"Not every graph is connected" ?

$\neg\forall_x\ (Graph(x) \rightarrow Connected(x))$

$\exists_x\ (Graph(x) \wedge \neg Connected(x))$

$\neg\forall_x\ (\neg Graph(x) \vee Connected(x))$

Q Consider the statement

“Tigers and lions attack if they are hungry or threatened”

$\forall_x[(tiger(x) \lor lion(x)) \rightarrow (hungry(x) \lor threatened(x)) \rightarrow attacks(x)]$

Q Consider the statement

Every teacher is liked by some student

$\forall_{(x)}$ [teacher (x) → $\exists_{(y)}$[student (y) ^ likes (y, x)]]

Q Consider the statement

Some boys in the class are taller than all the girls

$(\exists_x)\ (boy(x) \wedge (\forall_y)\ (girl(y) \rightarrow taller\ (x, y)))$

Q Convert the following two statements in quantified expressions of predicate logic

- For every number there is a number greater than that number.

P(x) : x is a number

Q(y) : y is a number greater than x

$\forall_x (P(x) \rightarrow Q(y))$

- Sum of every two integer is an integer.

P(x) : x is a integer

S(x) : x is sum of integer

$\forall_x (S(x) \rightarrow P(x))$

- Not Every man is perfect.

P(x) : x is perfect man

$\sim \forall_x(P(x))$

- There is no student in the class who knows Spanish and German.

P(x) : x is a student

L(x) : x knows Spanish and German

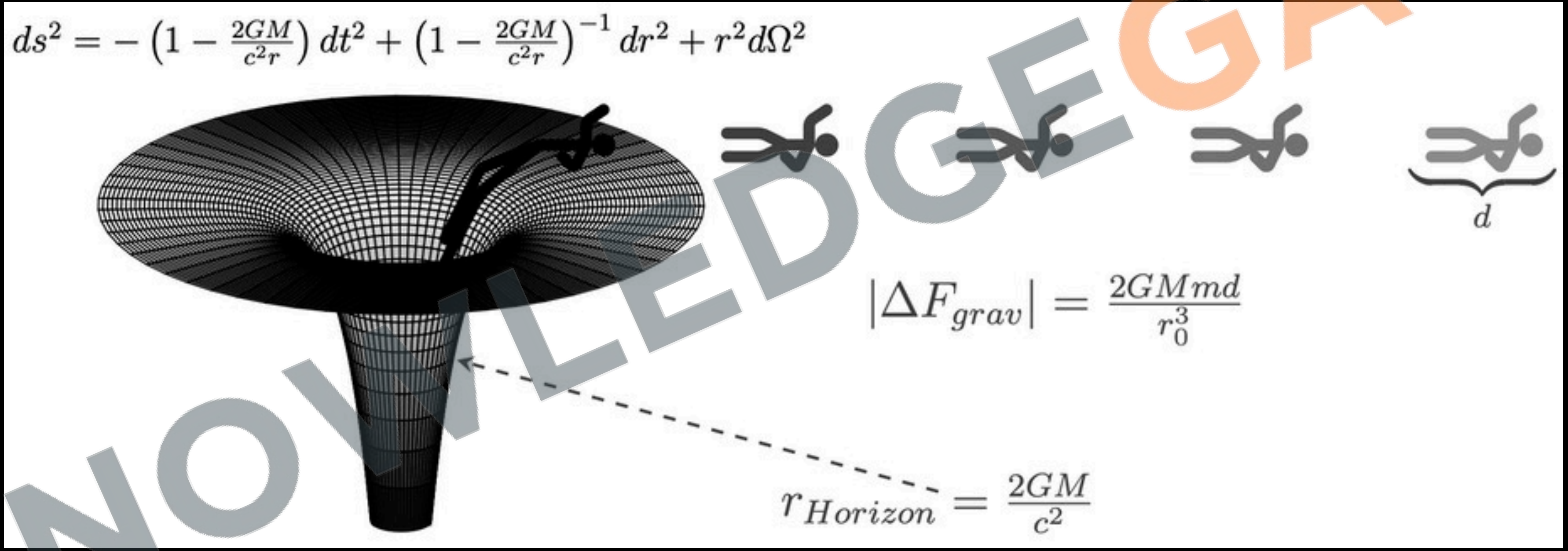
$\exists_x(P(x) \vee L(x))$

Chapter-6 (Algebraic Structures)

Group Theory

- Group theory is very important mathematical tool which is used in a number of areas in research and application. Using group theory, we can estimate the strength of a set with respect to an operator.

- This idea will further help us in research field to identify the correct mathematical system to work in a particular research area. E.g. can we use natural numbers in complex problem area like soft computing or studying black holes.



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- Now we will directly study some of the basic set related properties and will define some structure of set and operator based on the properties and will check those properties on basics number systems like natural numbers, integers, real numbers etc.

1. **Closure property: -** Consider a non-empty set A and a binary operation * on A. A is said to be closed with respect to *, if ∀ a, b ∈ A, then a*b ∈ A.

2. **Algebraic Structure: -** A non-empty set A is said to be an algebraic structure with respect to a binary operation *, if A satisfy closure property with respect to *.

		Algebraic Structure
1	(N, +)	
2	(N, -)	
3	(N, /)	
4	(N, x)	
5	(Z, +)	
6	(Z, -)	
7	(Z, /)	
8	(Z, x)	
9	(R, +)	
10	(R, -)	
11	(R, /)	
12	(R, x)	
13	(M, +)	
14	(M, x)	
15	(E, +)	
16	(E, x)	
17	(O, +)	
18	(O, x)	
19	(R-0, x)	
20	(R-0, /)	
21	(Non-Singular Matrix, x)	

1. **Associative property: -** Consider a non-empty set A and a binary operation * on A. A is said to be associative with respect to *, if ∀ a, b, c ∈ A, then (a*b) *c = a*(b*c)

2. **Semi-Group: -** A non-empty set A is said to be a Semi-group with respect to a binary operation *, if A satisfy closure, Associative property with respect to *.

		Algebraic Structure	Semi Group
1	(N, +)	Y	
2	(N, -)	N	
3	(N, /)	N	
4	(N, x)	Y	
5	(Z, +)	Y	
6	(Z, -)	Y	
7	(Z, /)	N	
8	(Z, x)	Y	
9	(R, +)	Y	
10	(R, -)	Y	
11	(R, /)	N	
12	(R, x)	Y	
13	(M, +)	Y	
14	(M, x)	Y	
15	(E, +)	Y	
16	(E, x)	Y	
17	(O, +)	N	
18	(O, x)	Y	
19	(R-0, x)	Y	
20	(R-0, /)	Y	
21	(Non-Singular Matrix, x)	Y	

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1. **Identity property: -** Consider a non-empty set A and a binary operation * on A. A is said to satisfy identity property with respect to *, if ∀ a ∈ A, there must be unique e ∈ A, such that
   a*e = e*a = a

2. There is exactly one Identity element in the set and will be same for all element in the set.

3. **Monoid: -** A non-empty set A is said to be a Monoid with respect to a binary operation *, if A satisfy closure, Associative, identity property with respect to *.

		Algebraic Structure	Semi Group	Monoid
1	(N, +)	Y	Y	
2	(N, -)	N	N	
3	(N, /)	N	N	
4	(N, x)	Y	Y	
5	(Z, +)	Y	Y	
6	(Z, -)	Y	N	
7	(Z, /)	N	N	
8	(Z, x)	Y	Y	
9	(R, +)	Y	Y	
10	(R, -)	Y	N	
11	(R, /)	N	N	
12	(R, x)	Y	Y	
13	(M, +)	Y	Y	
14	(M, x)	Y	Y	
15	(E, +)	Y	Y	
16	(E, x)	Y	Y	
17	(O, +)	N	N	
18	(O, x)	Y	Y	
19	(R-0, x)	Y	Y	
20	(R-0, /)	Y	N	
21	(Non-Singular Matrix, x)	Y	Y	

1. **Inverse property: -** Consider a non-empty set A and a binary operation * on A. A is said to satisfy inverse property with respect to *, if ∀ a ∈ A, there must be unique element a^{-1} ∈ A, such that $a * a^{-1} = a^{-1} * a = e$

2. Every element has a exactly one unique inverse which is also present in the same set.

3. If a is the inverse of b, then b will be invers one a.

4. No two elements can have the same inverse

5. Identity element is its own inverse.

6. **Group: -** A non-empty set A is said to be a group with respect to a binary operation *, if A satisfy closure, Associative, identity, inverse property with respect to *.

		AS	Semi Group	Monoid	Group
1	(N, +)	Y	Y	N	
2	(N, -)	N	N	N	
3	(N, /)	N	N	N	
4	(N, x)	Y	Y	Y	
5	(Z, +)	Y	Y	Y	
6	(Z, -)	Y	N	N	
7	(Z, /)	N	N	N	
8	(Z, x)	Y	Y	Y	
9	(R, +)	Y	Y	Y	
10	(R, -)	Y	N	N	
11	(R, /)	N	N	N	
12	(R, x)	Y	Y	Y	
13	(M, +)	Y	Y	Y	
14	(M, x)	Y	Y	Y	
15	(E, +)	Y	Y	Y	
16	(E, x)	Y	Y	N	
17	(O, +)	N	N	N	
18	(O, x)	Y	Y	Y	
19	(R-0, x)	Y	Y	Y	
20	(R-0, /)	Y	N	N	
21	(Non-Singular Matrix, x)	Y	Y	Y	

1. If the total number of elements in a group is even then there exists at least one element in the group who is the inverse of itself.

2. Some time it is also possible that every element is inverse of itself in a group.

3. In a group $(a * b)^{-1} = b^{-1} * a^{-1}$ for $\forall$ a, b $\in$ A

4. Cancelation law holds good

    1. $a * b = a * c \rightarrow b = c$

    2. $a * c = b * c \rightarrow a = b$

1. **Commutative property: -** Consider a non-empty set A and a binary operation * on A. A is said to satisfy commutative property with respect to *, if ∀ a, b ∈ A, such that a* b = b*a

2. **Abelian Group: -** A non-empty set A is said to be a group with respect to a binary operation *, if A satisfy closure, Associative, identity, inverse, commutative property with respect to *.

		AS	SG	Monoid	Group	Abelian Group
1	(N, +)	Y	Y	N	N	
2	(N, -)	N	N	N	N	
3	(N, /)	N	N	N	N	
4	(N, x)	Y	Y	Y	N	
5	(Z, +)	Y	Y	Y	Y	
6	(Z, -)	Y	N	N	N	
7	(Z, /)	N	N	N	N	
8	(Z, x)	Y	Y	Y	N	
9	(R, +)	Y	Y	Y	Y	
10	(R, -)	Y	N	N	N	
11	(R, /)	N	N	N	N	
12	(R, x)	Y	Y	Y	N	
13	(M, +)	Y	Y	Y	Y	
14	(M, x)	Y	Y	Y	N	
15	(E, +)	Y	Y	Y	Y	
16	(E, x)	Y	Y	N	N	
17	(O, +)	N	N	N	N	
18	(O, x)	Y	Y	Y	N	
19	(R-0, x)	Y	Y	Y	Y	
20	(R-0, /)	Y	N	N	N	
21	(Non-Singular Matrix, x)	Y	Y	Y	Y	

Q let A = {1, 3, 5, ……., ∞} and B = {2, 4, 6, ……., ∞}, what is the highest structure achieved by anyone of them?

1) (A,+) **2)** (A,*) **3)** (B,+) **4)** (B,*)

Q Consider a set of natural numbers N, with respect to *, such that a * b = a^b which structure it will satisfy?

Q let {p, q, r, s} be the set. A binary operation * is defined on the set and is given by the following table: Which of the following is true about the binary operation?

a) it is commutative but not associative

b) it is associative but not commutative

c) it is both associative and commutative

d) it is neither associative nor commutative

*	p	q	r	s
P	p	r	s	p
q	p	q	r	s
r	p	q	p	r
S	p	q	q	q

Q which of the following is not a group?

a) $\{\ldots\ldots. -6, -4, -2, 0, 2, 4, 6, \ldots\ldots\}, +$

b) $\{\ldots. -3k, -2k, -k, 0, k, 2k, 3k, \ldots.\}, + \; [k \in Z]$

c) $\{2^n, n \in Z\}, *$

d) set of complex number, *

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Q Consider the set of all integers(Z) with the operation defined as $m * n = m + n + 2,\ m, n \in Z$
if $(z, *)$ forms a group, then determine the identity element

a) 0 **b)** -1 **c)** -2 **d)** 2

Q Consider a set of positive rational number with respect to an operation *, such that a*b = (a.b)/3, it is known that the it is an abelian group, which of the following is not true?

a) identity element e = 3

b) inverse of a = 9/a

c) inverse of 2/3 = 6

d) inverse of 3 = 3

Q A binary operation α on a set of integers is defined as $x * y = x^2 + y^2$. Which one of the following statements is TRUE about *?

(A) Commutative but not associative

(B) Both commutative and associative

(C) Associative but not commutative

(D) Neither commutative nor associative

Q Which one of the following in NOT necessarily a property of a Group?

(A) Commutativity

(B) Associativity

(C) Existence of inverse for every element

(D) Existence of identity

- **Finite Group: -** A group with finite number of elements is called a finite group.

- **Order of group: -** Order of a group is denoted by O(G) = no of elements in G
  - If there is only one element in the Group, it must be an identity element.

Q Check out which of the following is a finite group?

1- {0}, +

+	0
0	

2- {0}, *

*	0
0	

3- {1}, +

+	1
1	

4- {1}, *

*	1
1	

5- {0,1}, +

+	0	1
0		
1		

6- {0,1}, *

*	0	1
0		
1		

7- {-1, 0, 1}, +

+	-1	0	1
-1			
0			
+1			

8- {-1, 0, 1}, *

*	-1	0	1
-1			
0			
1			

Q Check out which of the following is a finite group?

9- {-1, 1}, +

+	-1	1
-1		
1		

10- {-1, 1}, *

*	-1	1
-1		
1		

11- {-2, -1, 0, 1, 2}, +

+	-2	-1	0	1	2
-2					
-1					
0					
1					
2					

Q Check out which of the following is a finite group?

12- {-2, -1, 0, 1, 2}, *

*	-2	-1	0	1	2
-2					
-1					
0					
1					
2					

13- $\{1, \omega, \omega^2\}$, *

*	1	ω	ω^2
1			
ω			
ω^2			

14- {-1, 1, i, -i}, *

*	-1	1	i	-i
-1				
1				
i				
-i				

1. **Conclusion: -** it is very difficult to design finite group as with number greater than 2 closure property fails with simple addition and multiplication operation.

2. So we will try to develop new modified addition and multiplication operators with which closure and other properties can be satisfied.

- **Addition modulo: -** addition modulo is a binary operator denoted by $+_m$ such that

- $a +_m b = a + b$ if $(a + b < m)$

- $a +_m b = a + b - m$ if $(a + b >= m)$

$\{0,1,2,3\}, +_4$

$+_4$	0	1	2	3
0				
1				
2				
3				

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- **Multiplication modulo:** - Multiplication modulo is a binary operator denoted by $*_m$ such that

- $a *_m b = a * b$ if $(a * b < m)$

- $a *_m b = (a * b) \% m$ if $(a * b >= m)$

$\{1,2,3,4\}, *_5$

$*_5$	1	2	3	4
1				
2				
3				
4				

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Q Check out which of the following is a group?

1- $\{0,1,2,3\}, +_4$

$+_4$	**0**	**1**	**2**	**3**
0				
1				
2				
3				

2- $\{0,1,2,3\}, *_4$

$*_4$	**0**	**1**	**2**	**3**
0				
1				
2				
3				

3- $\{1,2,3\}, +_4$

$+_4$	**1**	**2**	**3**
1			
2			
3			

4- $\{1,2,3\}, *_4$

$*_4$	**1**	**2**	**3**
1			
2			
3			

5- {0,1,2,3,4}, $+_5$

$+_5$	0	1	2	3	4
0					
1					
2					
3					
4					

6- {0,1,2,3,4}, $*_5$

$*_5$	0	1	2	3	4
0					
1					
2					
3					
4					

7- {1,2,3,4}, $+_5$

$+_5$	1	2	3	4
1				
2				
3				
4				

8- {1,2,3,4}, $*_5$

$*_5$	1	2	3	4
1				
2				
3				
4				

9- {0,1,2,3,4,5,6}, $+_7$

$+_7$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

10- {0,1,2,3,4,5,6}, $*_7$

$*_7$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

11- {1,2,3,4,5,6}, $+_7$

$+_7$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

12- {1,2,3,4,5,6}, $*_7$

$*_7$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

13- {1,3,5,7}, $*_8$

$*_8$	1	3	5	7
1				
3				
5				
7				

14- {1,2,4,7,8,11,13,14}, $*_{15}$

$*_{15}$	1	2	4	7	8	11	13	14
1								
2								
4								
7								
8								
11								
13								
14								

15- $\{1,2,3, 4\ldots\ldots, p-1\}, *_p$

16- $\{0,1,2,3, 4\ldots\ldots, p-1\}, *_p$

17- $\{1,2,3, 4\ldots\ldots, p-1\}, +_p$

18- $\{0,1,2,3, 4\ldots\ldots, p-1\}, +_p$

Q Let s = set of all integers. A binary operation * is defined by
$a * b = a + b + 3$
consider the following statements

S_1: (S,*) is a group

S_2: -3 is identity element of (S, *)

S_3: the inverse of -6 is 0

which of the following are true

a) Only S_1 and S_2
b) Only S_2 and S_3
c) Only S_1 and S_3
d) Only S_1 ,S_2 and S_3

Q $\{0,1,2,3,4,5\}, +_6$ is a group which of the following is not true?

a) $1^{-1} = 5$ **b)** $2^{-1} = 4$ **c)** $3^{-1} = 6$ **d)** $0^{-1} = 0$

$+_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

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Q {1,2,3,4,5,6}, $*_7$ is a group which of the following is not true?

a) $1^{-1} = 1$ **b)** $2^{-1} = 4$ **c)** $3^{-1} = 5$ **d)** $6^{-1} = 6$

$*_7$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Q $\{1,3,5,7\}, *_8$ is a group which of the following is not true?

a) $1^{-1} = 1$ **b)** $3^{-1} = 3$ **c)** $5^{-1} = 5$ **d)** $7^{-1} = 7$

$*_8$	1	3	5	7
1				
3				
5				
7				

Q {1,2,4,7,8,11,13,14}, $*_{15}$ is a group which of the following is not true?

a) $2^{-1} = 8$ **b)** $4^{-1} = 4$ **c)** $7^{-1} = 13$ **d)** $11^{-1} = 14$

$*_{15}$	1	2	4	7	8	11	13	14
1								
2								
4								
7								
8								
11								
13								
14								

Sub Group

1. The subset of a group may or may not be a group.

2. When the subset of a group is also a group then it is called sub group.

3. The identity element of a group and its sub group is always same.

4. Union of two subgroup may or may not be a subgroup.

5. Intersection of two subgroup is always a subgroup.

6. Lagrange's theorem: - the order of a group is always exactly divisible by the order of a sub group.

Q Consider a group G = {1,3,5,7}, $*_8$ which of the following sub set of this set does not form is sub group?

a) {0,1} **b)** {1,3} **c)** {1,5} **d)** {1,7} **e)** {1,3,7}

$*_8$	0	1
0		
1		

$*_8$	1	3
1		
3		

$*_8$	1	5
1		
5		

$*_8$	1	7
1		
7		

$*_8$	1	3	7
1			
3			
7			

Q Let G be a group with 15 elements. Let L be a subgroup of G. It is known that L != G and that the size of L is at least 4. The size of L is ____________.

(A) 3 **(B)** 5 **(C)** 7 **(D)** 9

Q Let G be a finite group on 84 elements. The size of a largest possible proper subgroup of G is ________ .

Order of an element: - (A, *) be a group, then ∀ a ∈ A, order of a is denoted by O(a).

1. O(a) =n (smallest positive integer), such that $a^n = e$
2. Order of identity element is always one.
3. Order of an element and its inverse is always same.
4. Order of an element in an infinite group does not exist or infinite expect identity.

Q consider a group {0,1,2,3}, $+_4$ and find the order of each element?

$+_4$	0	1	2	3
0				
1				
2				
3				

Q consider a set on cube root of unity $\{1, \omega, \omega^2\}$, * and find the order of each element?

*	1	ω	ω^2
1			
ω			
ω^2			

Q consider a set on forth root of unity {-1, 1, I, -i}, * and find the order of each element?

*	-1	1	i	-i
-1				
1				
i				
-i				

Generating element or Generator: - A element ‘a’ is said to be a generating element, if every element of A is an integral power of a, i.e. every element of A can be represented using power of a.

$A = \{a^1, a^2, a^3, a^4, a^5....\}$

Cyclic group: - A group (A, *) is said to be a cyclic group if it contains at least one generator.

1. In a cyclic group if an element is a generator than its inverse will also be a generator.

2. The order of a cyclic group is always the order of the generating element of G.

Q Show that G = {1,2,4,5,7,8}, $+_{15}$ is a cyclic group. How many generators are there?

$*_{15}$	1	2	4	5	7	8
1						
2						
4						
5						
7						
8						

Q consider a group {1,2,4,7,8,11,13,14}, $*_{15}$ and find the order of each element?

$*_{15}$	1	2	4	7	8	11	13	14
1								
2								
4								
7								
8								
11								
13								
14								

Q For the composition table of a cyclic group shown below

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Which one of the following choices is correct?

(A) a, b are generators

(B) b, c are generators

(C) c, d are generators

(D) d, a are generators

Number of generators

Lagrange's theorem: - let A be a cyclic group of order n, number of Generator in A is denoted by $\phi(n) = \{n(p_1-1)(p_2-1)(p_3-1) \ldots\ldots\ldots (p_k-1)\} / (p_1p_2p_3\ldots\ldots\ldots p_k)$

Q let G be a cyclic group, $O(G) = 8$, number of generators in G =?

Q let G be a cyclic group, O(G) = 12, number of generators in G =?

Q let G be a cyclic group, $O(G) = 70$, number of generators in G =?

Q let G be a cyclic group, O(G) = 23100, number of generators in G =?

Coset

Let H be a subgroup of a group G and $a \in G$, then the set
$aH = \{ah \mid h \in H\}$ is called a left coset of H in G
and $Ha = \{ha \mid h \in H\}$ is called a right coset of H in G.

By this definition, it is clear that corresponding to every element of G, we have a left coset and right coset of H in G. It is obvious that

$$aH \subset G, Ha \subset G, \forall a \in G$$

Further we may note that

$$eH = H = He$$

i.e., the left and the right cosets of H corresponding to the identity e coincide with H. Hence H itself is a left as well as a right coset of H in G.

Let G= (Z, +) and H = 2Z = {0, ± 2, ± 4,....}

then the right cosets of H in G are :

H+ 0={0, ±2, ±4, ...} = H

H + 1={0 + 1, ± 2+1, ± 4+1, ...}

H + 2={0 + 2, ± 2+2, ± 4+2, ...} etc.

It can be easily observed that any right coset and its corresponding left coset are equal i.e.,

H+ 1=1 + H, H + 2=2 + H etc.

Again

H + 3 ={.., - 6, -3, 0, 3, 6, 9, 12,....} = H

H + 4 = {..., - 5, - 2, 1, 4, 7, 10, 13,..} = H + 1 etc.

Similarly H + 5 = H + 2, H + 6 = H

Thus H has only two distinct cosets H, H + 1 in Z.

Find all the cosets of 3Z in the group (Z, +).

H = 3Z = {.....6, -3, 0, 3, 6, ...}

The following distinct cosets of H in G are obtained :

0 + H = H + 0 = {..., -6, -3, 0, 3, 6, ...} = H

1 + H = H + 1 = {.., - 5, -2, 1, 4, 7, ...}

2 + H = H + 2 = {.....-4, - 1, 2, 5, 8,..}

3 + H = H + 3 = 0 + H = H + 0

4 + H = H + 4 = 1 + H = H + 1

5 + H = H + 5 = 2 + H = H + 2

6 + H = H + 6 = 0 + H = H + 0

three cosets of H in G are H, H+1, H+2

so n cosets of nZ in (Z,+)

Find all the cosets of $H = \{0, 3\}$ in the group $(Z_6, +_6)$.

$G = \{0, 1, 2, 3, 4, 5\}$ $H = \{0, 3\}$, G

The following distinct cosets of H in G are obtained :

$0 + H = H + 0 = \{0, 3\} = H$

$1 + H = H + 1 = \{0 +_6 1, 3 +_6 1\} = \{1, 4\} = H + 1$

$2 + H = H + 2 = \{0 +_6 2, 3 +_6 2\} = \{2, 5\} = H + 2$

$3 + H = H + 3 = \{0 +_6 3, 3 +_6 3\} = \{3, 0\} = H$

$4 + H = H + 4 = \{0 +_6 4, 3 +_6 4\} = \{4, 1\} = H + 1$

$5 + H = H + 5 = \{0 +_6 5, 3 +_6 5\} = \{5, 2\} = H + 2$

$6 + H = H + 6 = \{0 +_6 6, 3 +_6 6\} = \{0, 3\} = H$

If we take union of all distinct cosets of G, we will get back G

Ring

- A ring is an algebraic system (R, +, •) where R is a non-empty set and + and • are two binary operations (which can be different from addition and multiplication) and if the following conditions are satisfied :
  - (R, +) is an abelian group.
  - (R, •) is semigroup
- The operation • is distributive over +.
  - $a \bullet (b + c) = (a \bullet b) + (a \bullet c)$
  - $(a + b) \bullet c = (a \bullet c) + (b \bullet c)$

- E.g. (Z, +, X) is a ring

Integral Domain

- A ring becomes integral domain, if it is a commutative ring with unity and without zero divisor
  - (D, +) is an abelian group.
  - (D, •) is semigroup with Commutative and With unity, Without zero divisor if a.b=0 then a=0 or b=0

- E.g. (Z, +, X) is a integral domain, (Q, +, X) is a integral domain, (R, +, X) is a integral domain, (C, +, X) is a integral domain, $(Z, +_3, X_3)$ is a integral domain

Field

- A field is an algebraic system (R, +, •) where R is a non-empty set and + and • are two binary operations (which can be different from addition and multiplication) and if the following conditions are satisfied :
  - (R, +) is an abelian group
  - (R, •) is an abelian group(inverse should exist for every non-zero element)
  - The operation • is distributive over +.
  - a • (b + c) = (a • b) + (a • c)
  - (a + b) • c = (a • c) + (b • c)

- E.g. (R, +, X) is a field, (Q, +, X) is a field, (Z, $+_3$, X_3) is a field

$\{0,1,2,3,4\}, +_5$

$+_5$	0	1	2	3	4
0					
1					
2					
3					
4					

$\{0,1,2,3,4\}, *_5$

$*_5$	0	1	2	3	4
0					
1					
2					
3					
4					

This is ring, integral domain and field

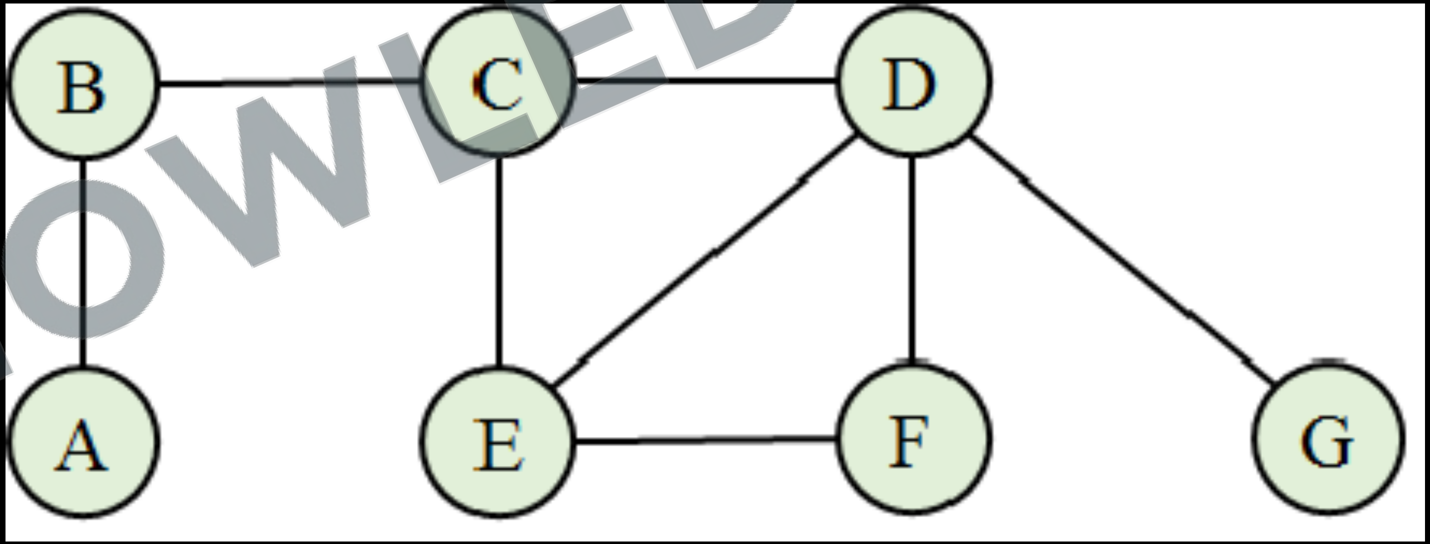
Chapter-7 (Graphs)

Graph Theory

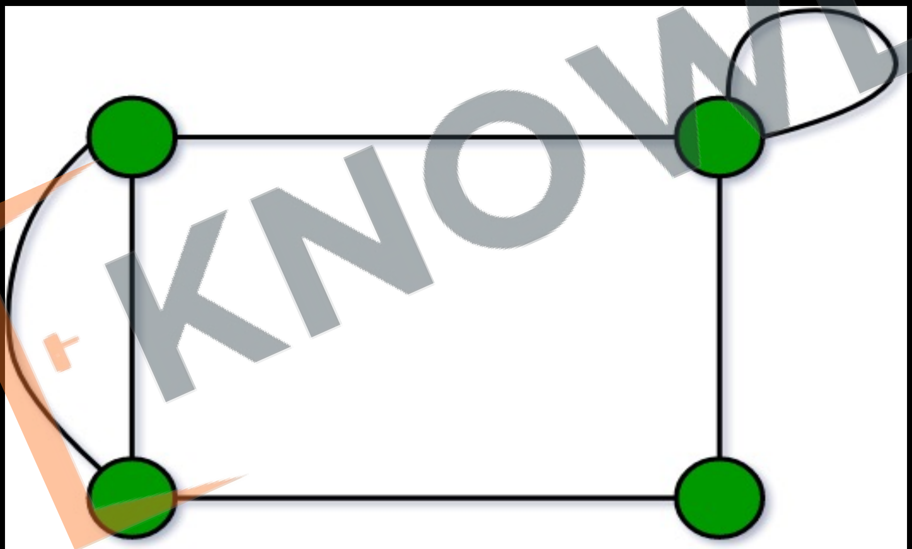
1. A graph G (V, E) consist of a set off objects $V = \{V_1, V_2, V_3 ...., V_N\}$ called vertices and another set $E = \{E_1, E_2, E_3, ..., E_n\}$ whose elements are called edges.

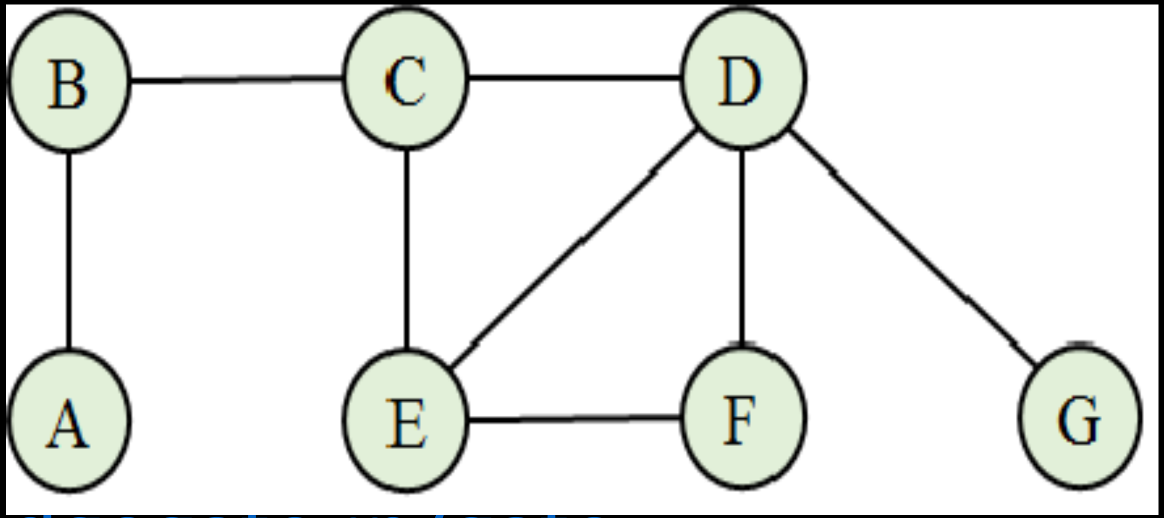
2. Each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

3. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.

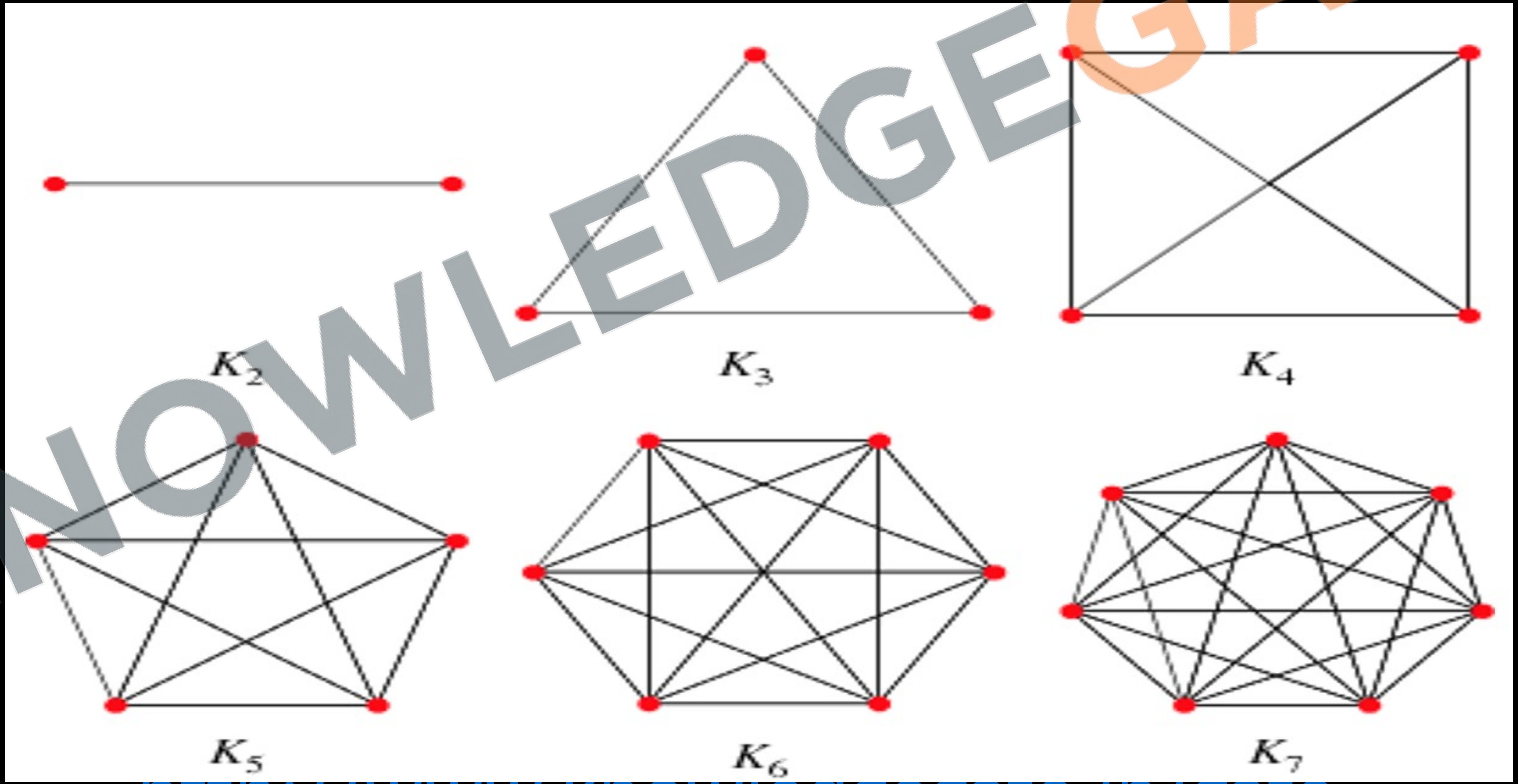


1. **Self-Loop:** Edge having the same vertex (v_i, v_i) as both its end vertices is called self-loop.
2. **Parallel Edge:** When more than one edge associated with a given pair of vertices such edges are referred as parallel edges.
3. **Adjacent Vertices:** If two vertices are joined by the same edges, they are called adjacent vertices.
4. **Adjacent Edges:** If two edges are incident on some vertex, they are called adjacent edges.





- **Complete or Full Graph:** In a simple graph there exist an edge between each and every pair of vertices i.e. every vertex are adjacent to each other, then the graph is said to be a complete graph, denoted by $K_{n.}$ Number of edges in a Complete graph is n(n-1)/2

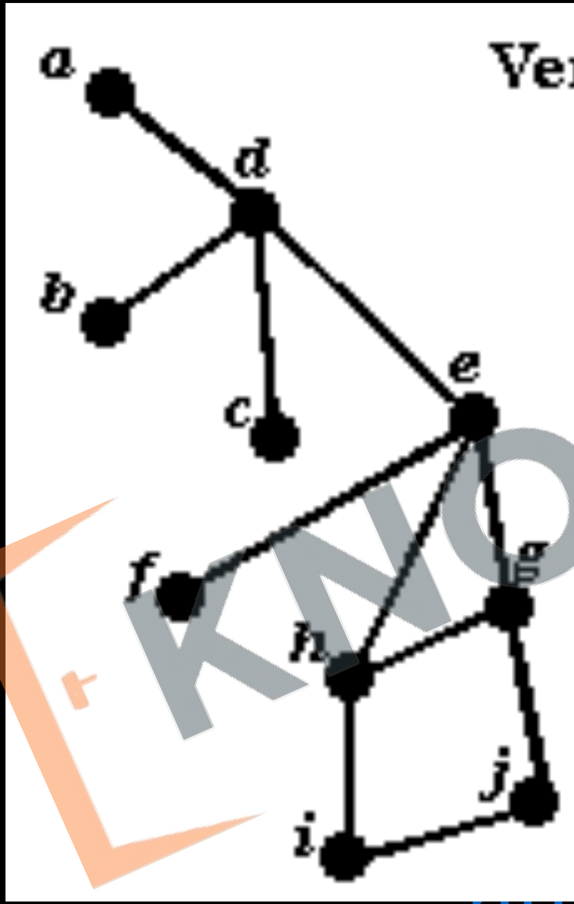


Degree

- **Degree of a Vertex:** The degree of a vertex in an undirected graph is the number of edges associated with it, denoted by deg(v_i).

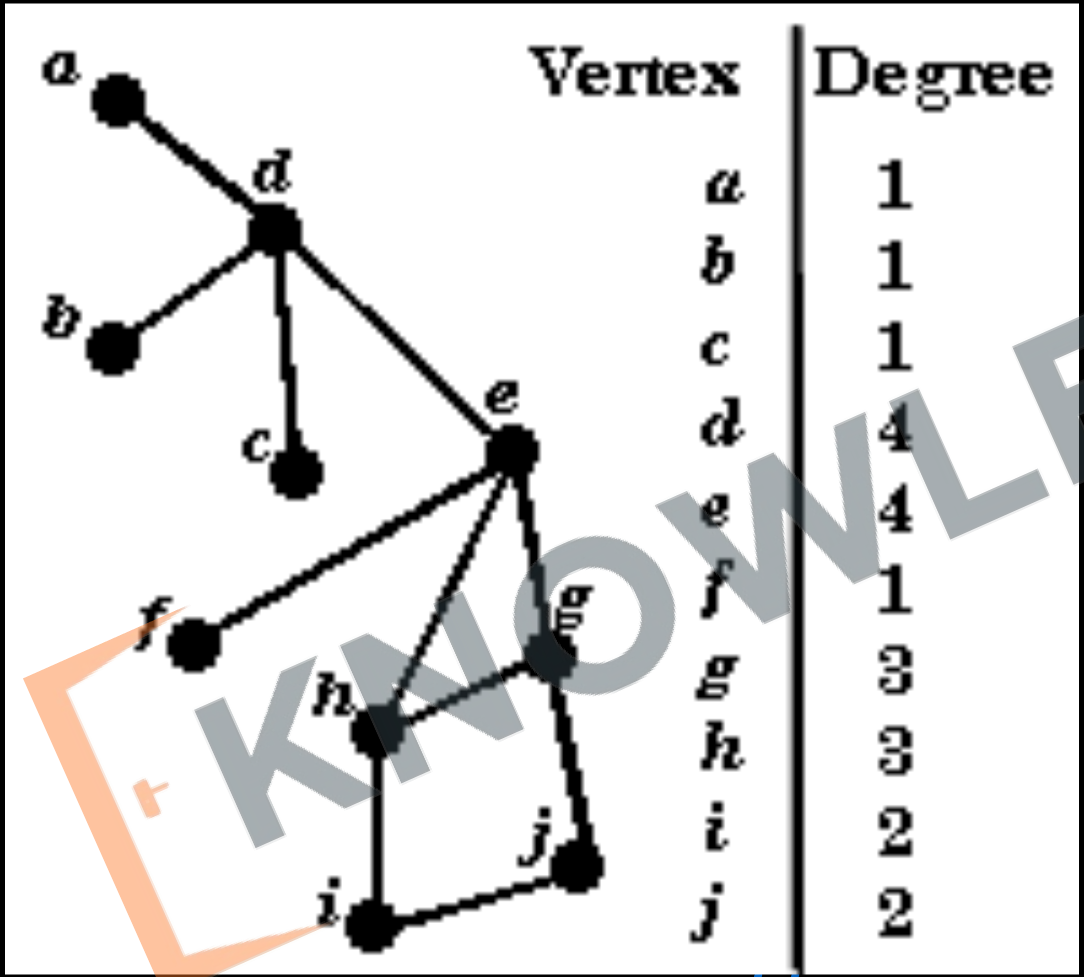
Vertex	Degree
a	1
b	1
c	1
d	4
e	4
f	1
g	3
h	3
i	2
j	2

- **Hand-shaking theorem:** - Since each edge contribute two degree in the graph, the sum of the degree of all vertices in G is twice the number of edges in g.
  - $\sum_{i=1}^{n} d(vi)$ = 2|E|



Vertex	Degree
a	1
b	1
c	1
d	4
e	4
f	1
g	3
h	3
i	2
j	2

- The number of vertices of odd degree in a graph is always even.
  - $\sum_{i=1}^{n} d(vi) = \sum_{even} d(vi) + \sum_{odd} d(vi)$



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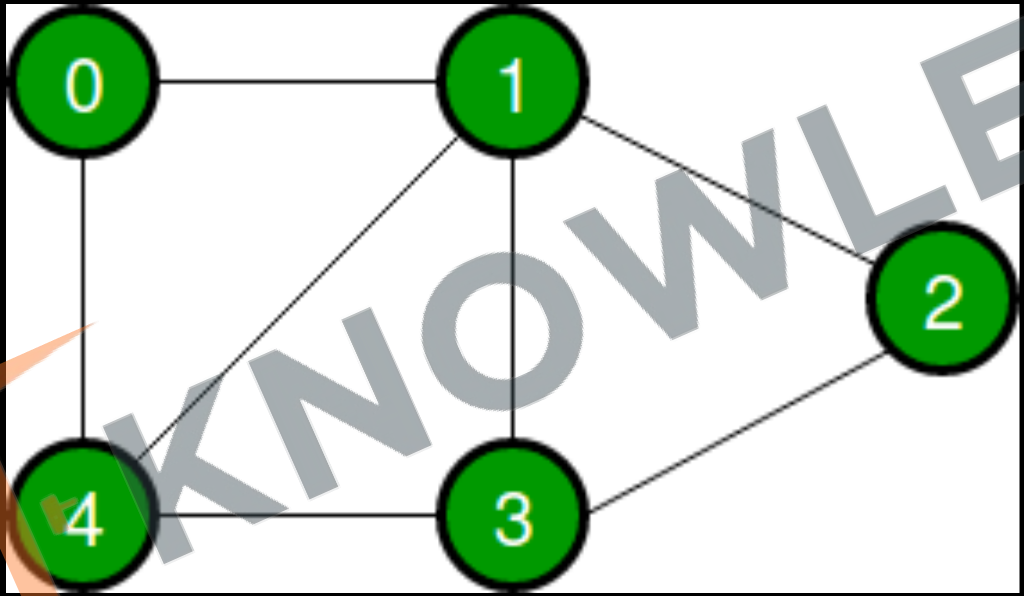
Representation of Graph in Memory

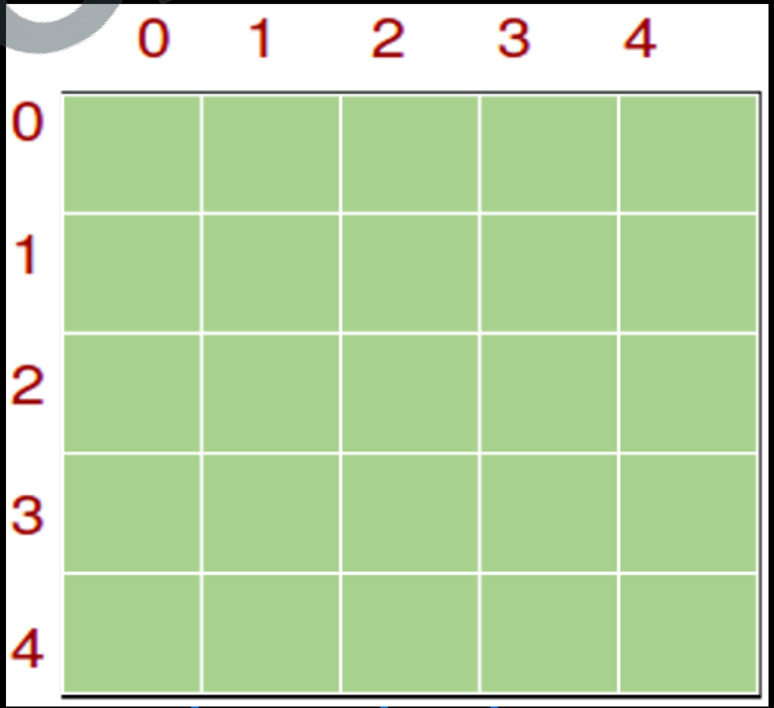
- Following two are the most commonly used representations of a graph.
  - Adjacency Matrix
  - Adjacency List
- There are other representations also like, Incidence Matrix and Incidence List. The choice of the graph representation is situation specific. It totally depends on the type of operations to be performed and ease of use.

- **Adjacency Matrix:** Adjacency Matrix is a 2D array of size V x V where V is the number of vertices in a graph. Let the 2D array be adj[][], a slot adj[i][j] = 1 indicates that there is an edge from vertex i to vertex j.

- Adjacency matrix for undirected graph is always symmetric.

- Adjacency Matrix is also used to represent weighted graphs. If adj[i][j] = w, then there is an edge from vertex i to vertex j with weight w.





Incidence Matrix

- **Representation of undirected graph** : Consider a undirected graph G = (V, E) which has n vertices and m edges all labelled. The incidence matrix I(G) = [bij], is then n x m matrix,
  - where bi,j=1 when edge ej is incident with vi
  - = 0 otherwise

- **Representation of directed graph** : The incidence matrix I(D) = [bij] of digraph D with n vertices and m edges is the n x m matrix in which.
  - Bi,j = 1 if arc j is directed away from vertex vi
  - = −1 if arc j is directed towards vertex vi
  - =0 otherwise.

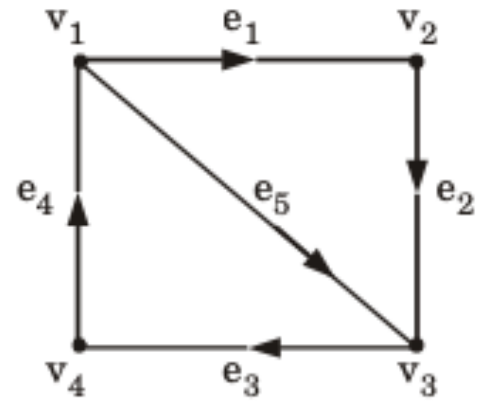
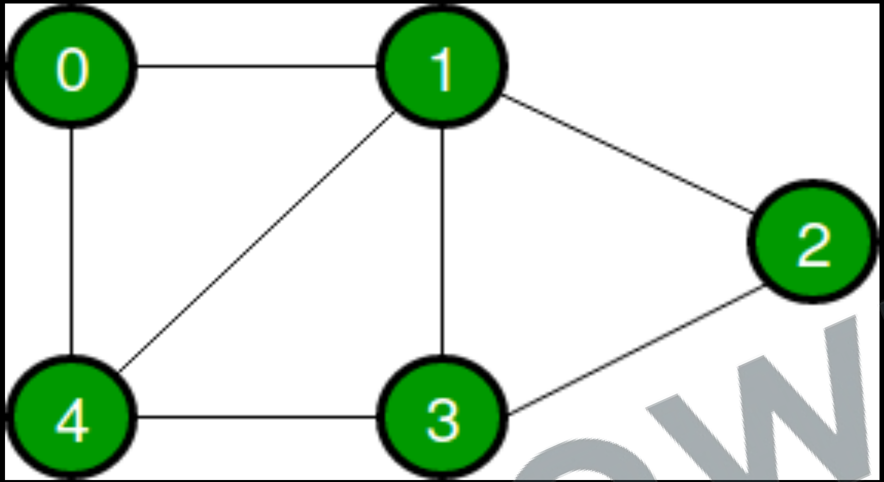


Fig. 4.3.3.

The incidence matrix of the digraph of Fig. 4.3.3 is

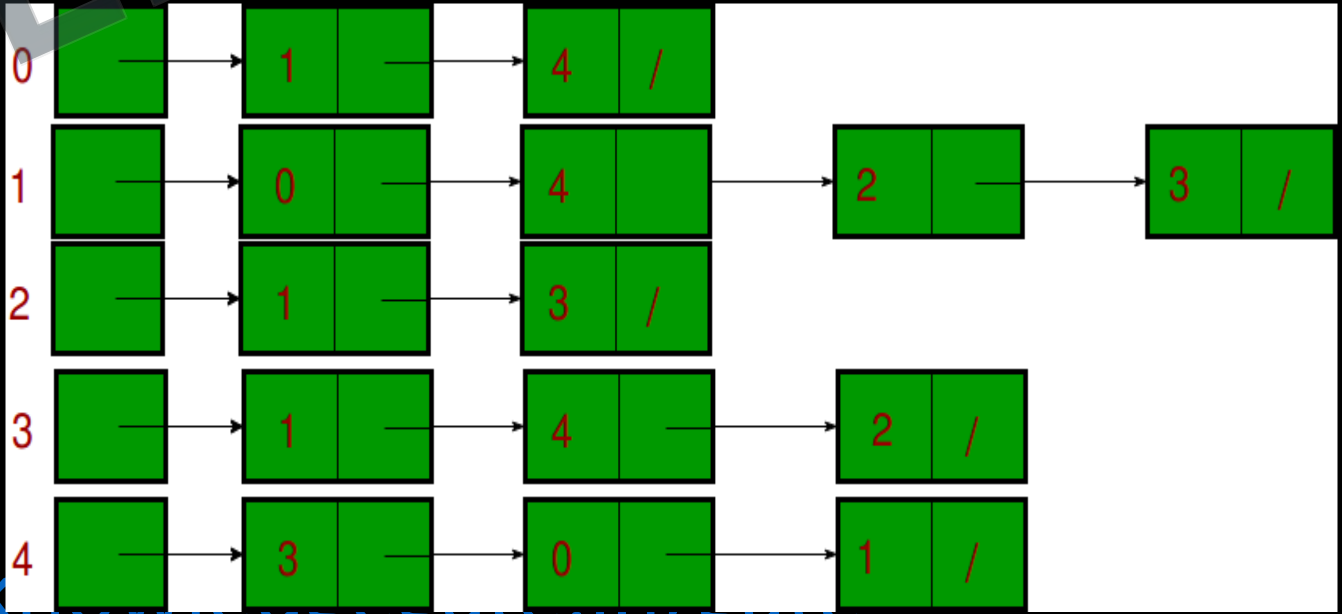
$$I(D) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

- **Adjacency List:** An array of lists is used. Size of the array is equal to the number of vertices. Let the array be array[]. An entry array[i] represents the list of vertices adjacent to the *i*th vertex. This representation can also be used to represent a weighted graph. The weights of edges can be represented as lists of pairs.



a. For non-weighted graph : | INFO | Adj-list |

b. For weighted graph : | Weight | INFO | Adj-list |



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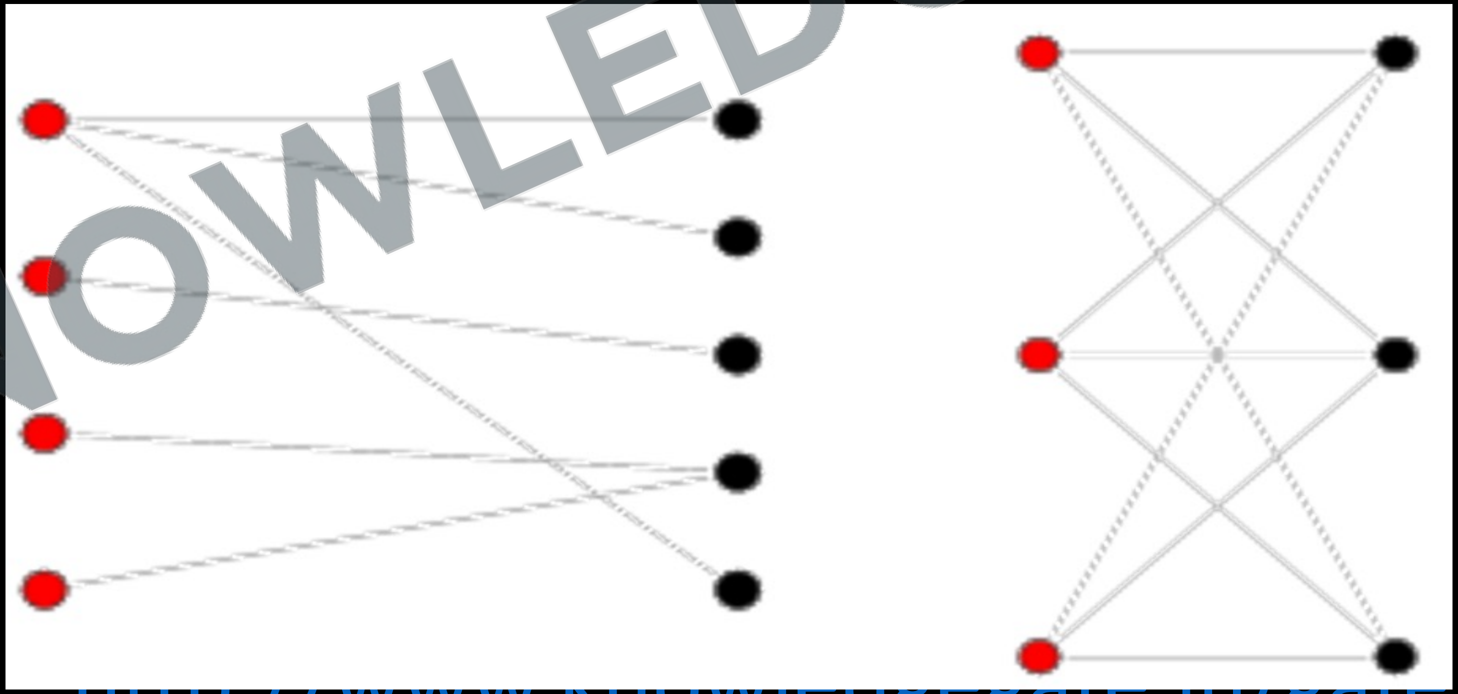
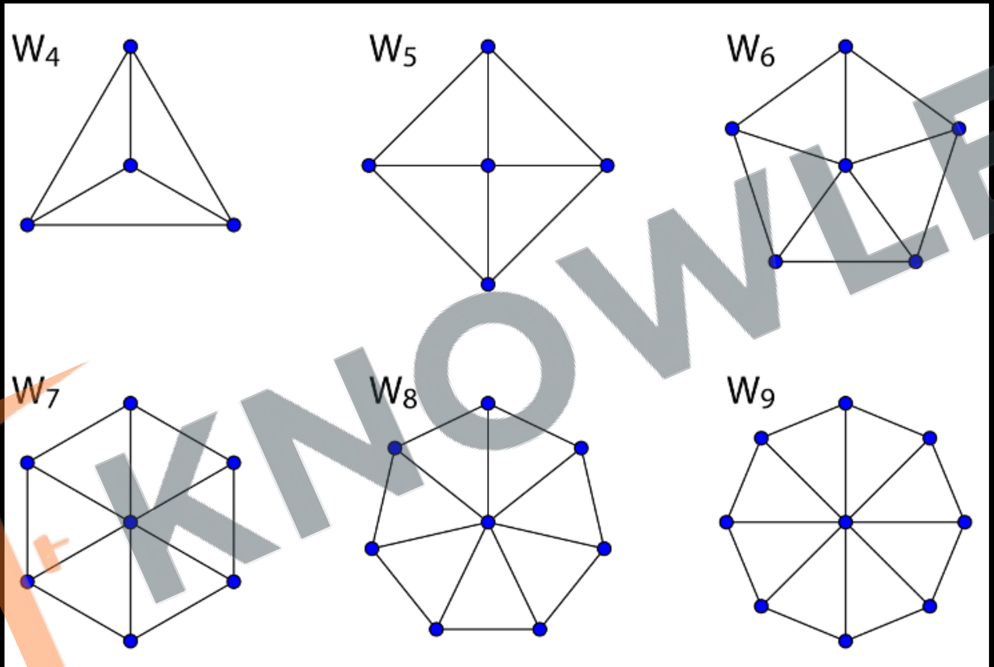
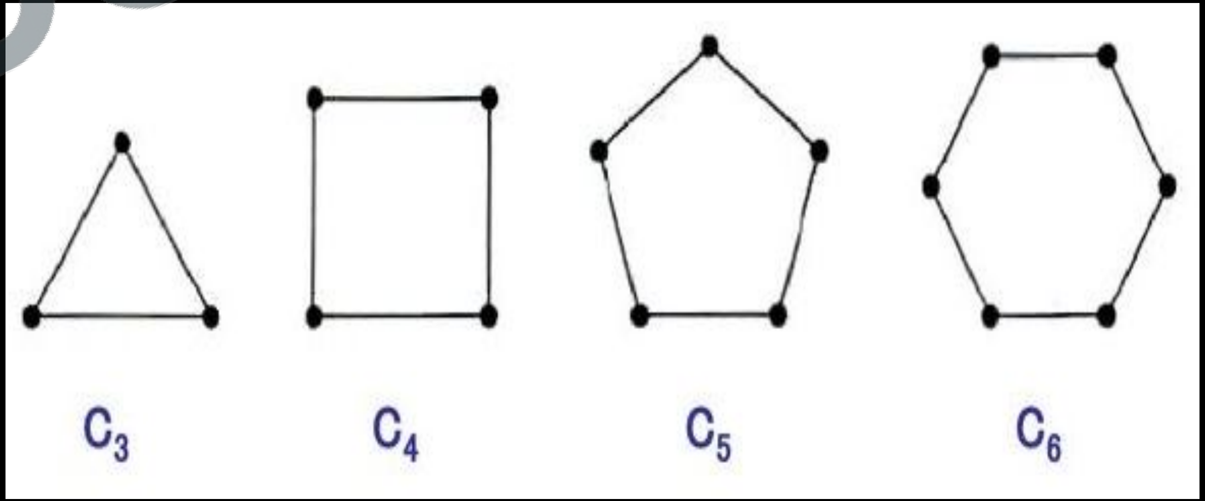
Some Popular Graph

1. **Bi-partite graph:** - A graph G(V, E) is called bi-partite graph if it's vertex set V(G) can be partitioned into two non-empty disjoint subset $V_1(G)$ and $V_2(G)$ in such a way that each edge e ϵ E(G) has it's one end point in $V_1(g)$ and other end point in $V_2(g)$. The partition $V = V_1 \cup V_2$ is called bipartition of G.
2. **Complete Bi-partite graph:** - A Bi-partite graph G(V, E) is called Complete bi-partite graph if every vertex in the first partition is connected to every vertex in the second partition, denoted by $K_{m,n}$.

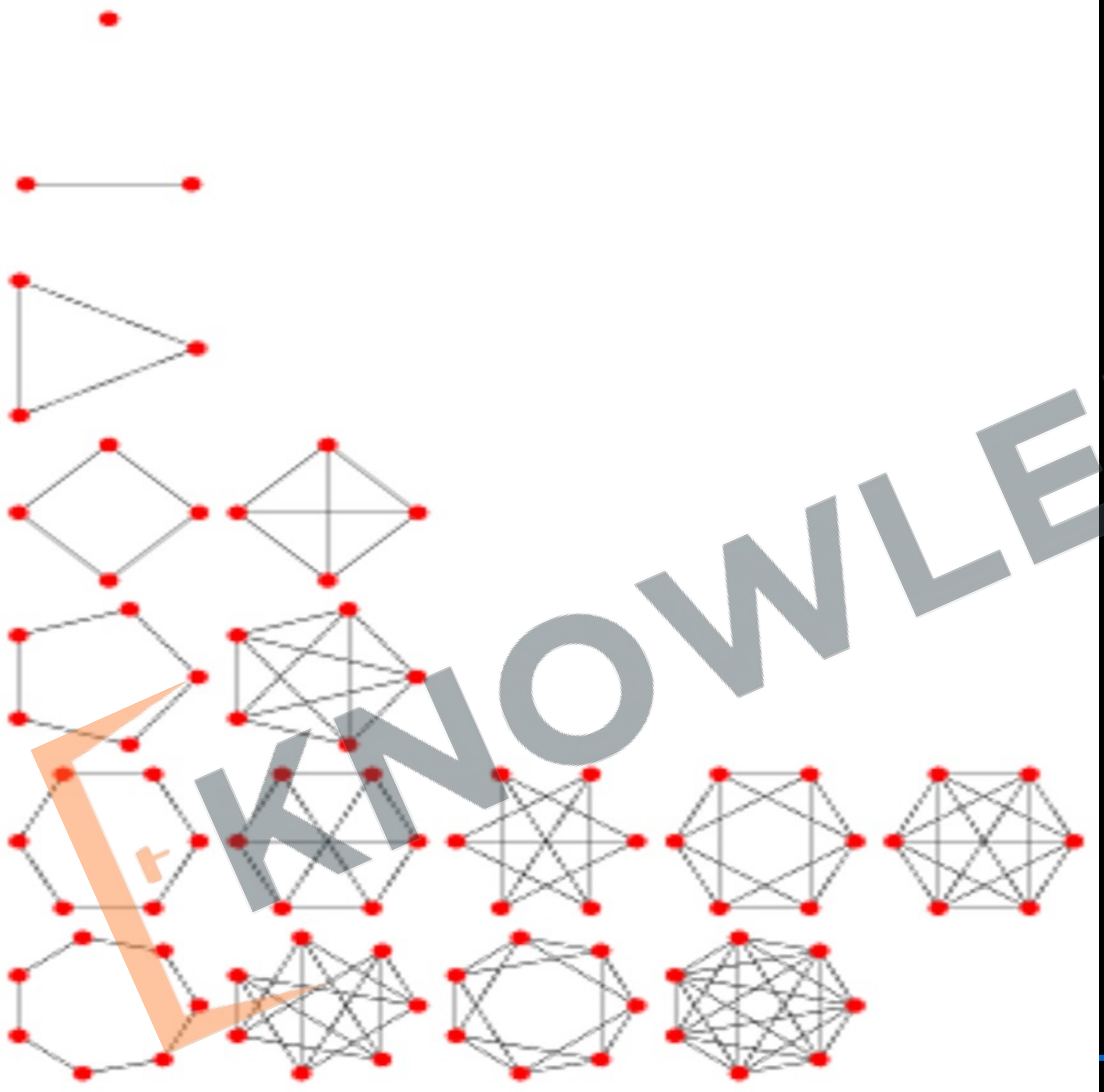


1. **Cycle Graph:** - A cycle graph or circular graph is a graph that consists of a single cycle, or in other words, some number of vertices (at least 3) connected in a closed chain. The cycle graph with n vertices is called C_n. The number of vertices in C_n equals the number of edges, and every vertex has degree 2; that is, every vertex has exactly two edges incident with it.
2. **Wheel graph:** - A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. Some authors write W_n to denote a wheel graph with n vertices ($n \geq 4$); other authors instead use W_n to denote a wheel graph with $n+1$ vertices ($n \geq 3$), which is formed by connecting a single vertex to all vertices of a cycle of length n.



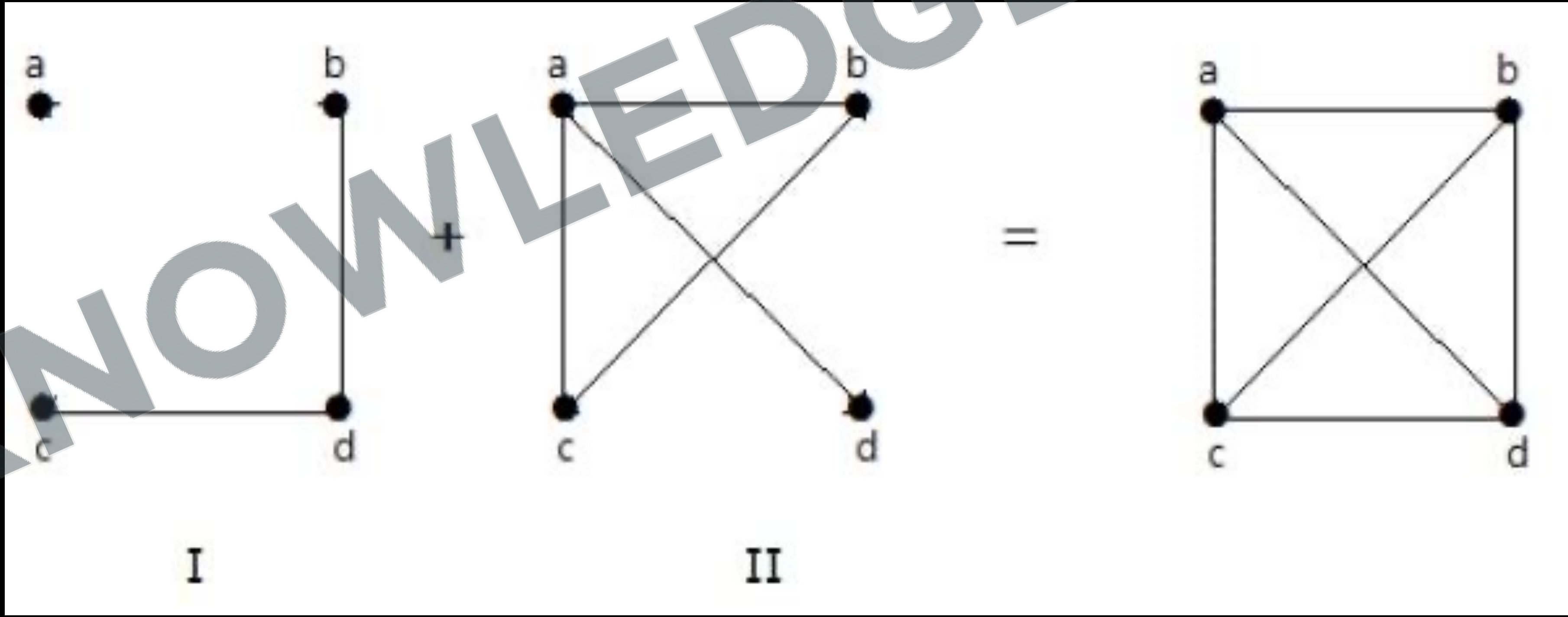


- **Regular graph:** - A graph in which all the vertices are of equal degree is called a regular graph. E.g. 2-regular graph, 3-regular graph.



Complement of a Graph

1. The complement of a simple graph G (V, E) is a graph G^c (V, E^c) on the same vertices set as of G, such that there will be an edge between two vertices u, v in G^c if ang only if there is no edge between u, v in G. i.e. two vertices of Gc are adjacent iff they are not adjacent in G.
2. $V(G) = V(G^c)$
3. $E(G^c) = \{(u, v) \mid (u, v) \notin E(G)\}$
4. $E(G^c) = E(K_n) - E(G)$



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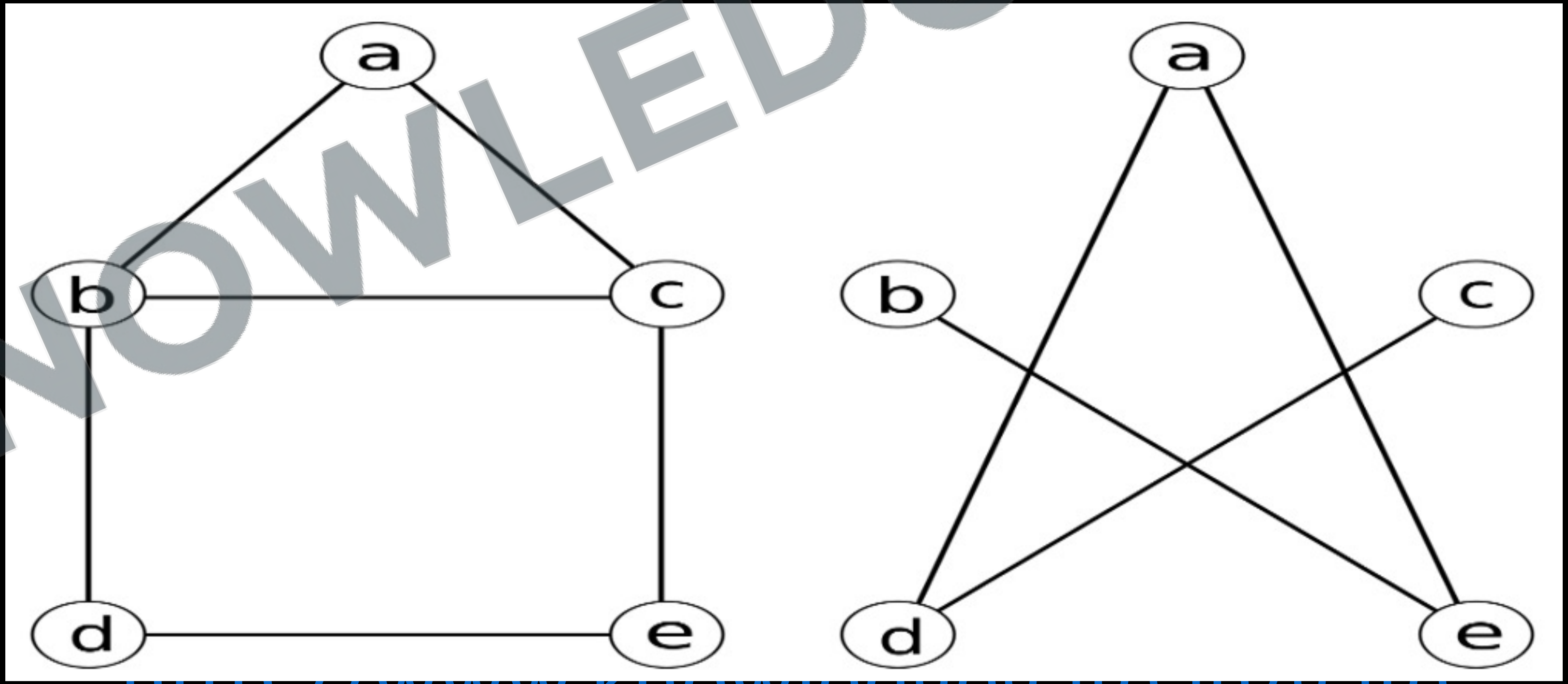
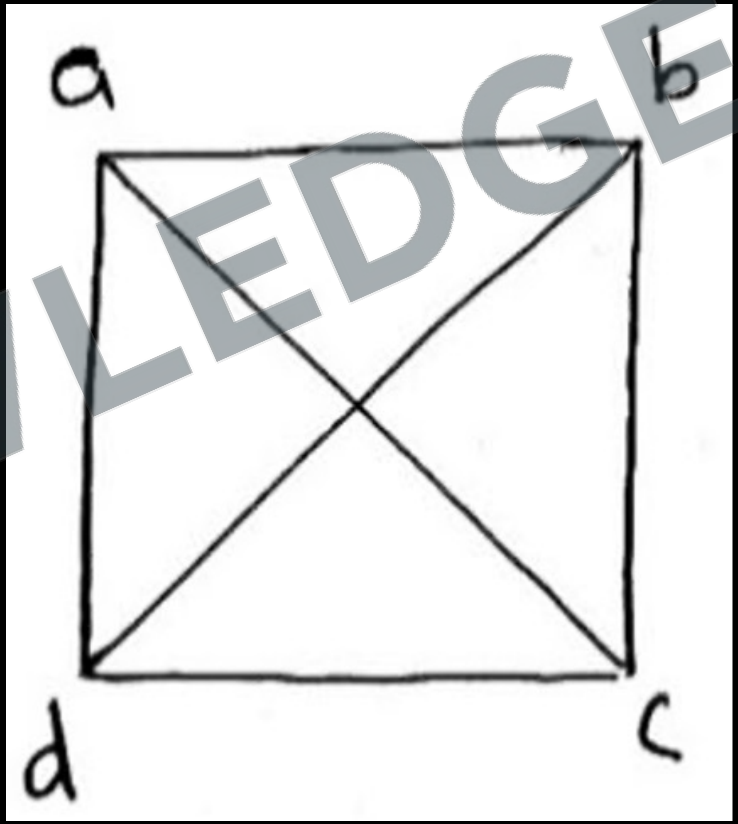
Properties

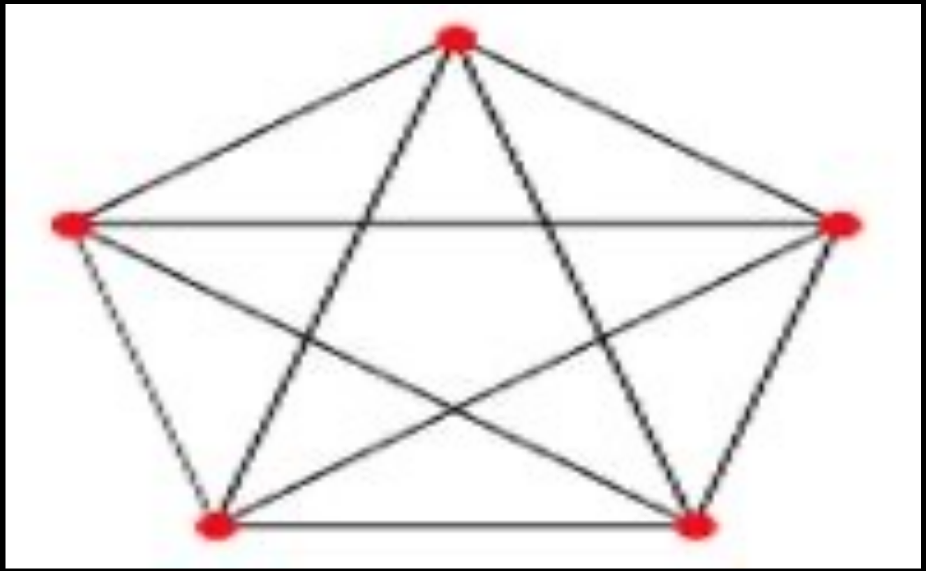
1. $G \cup G^c = K_n$
2. $G \cap G^c$ = null graph
3. $|E(G)| + |E(G^c)| = E(K_n) = n(n-1)/2$

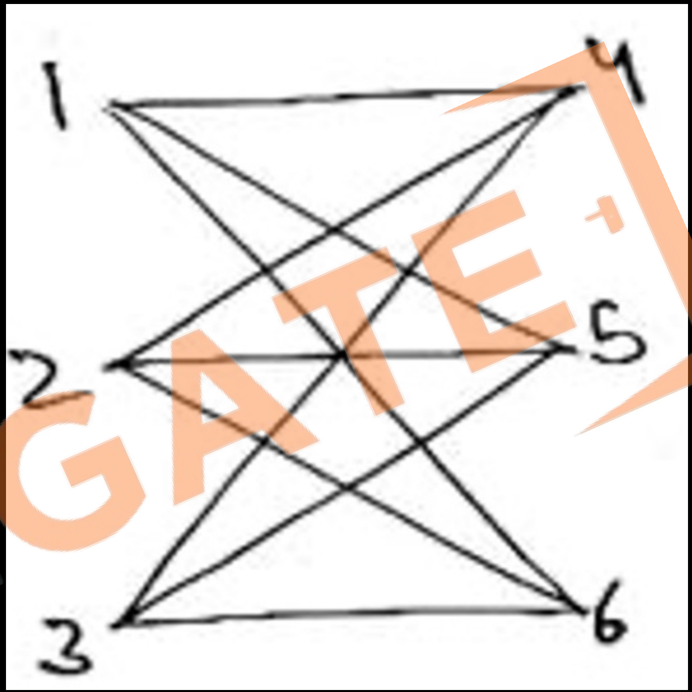


Planer Graph

Planer Graph: - A graph is called a planer graph if it can be drawn on a plan in such a way that no edges cross each other, otherwise it is called non-planer. Application: civil engineering, circuit designing



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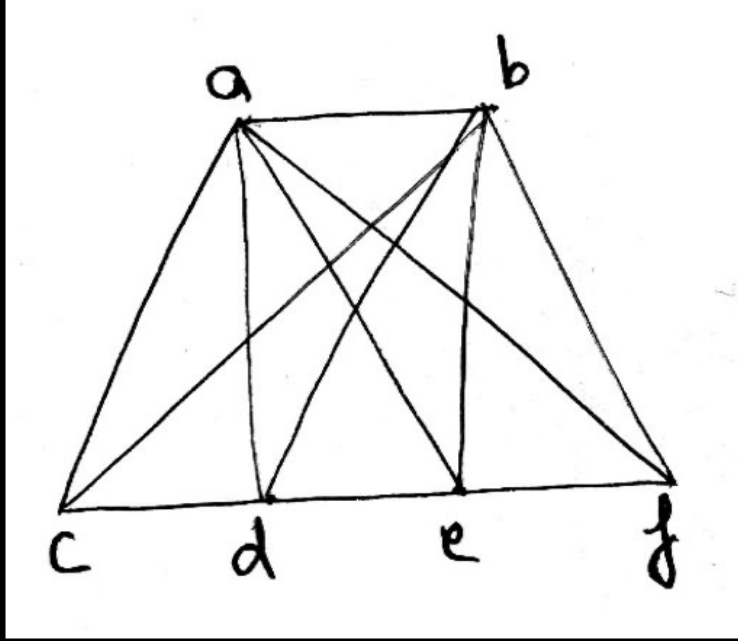
Simplest Non-Planer Graphs

1. **Kuratowski's case I:** - K_5
2. **Kuratowski's case II:** - $K_{3,3}$
3. Both are simplest non-planer graph
4. Both are regular graph
5. If we delete either an edge or a vertex from any of the graph, they will become planer

- **Kazimierz Kuratowski** (2 February 1896 – 18 June 1980) was a Polish mathematician and logician.



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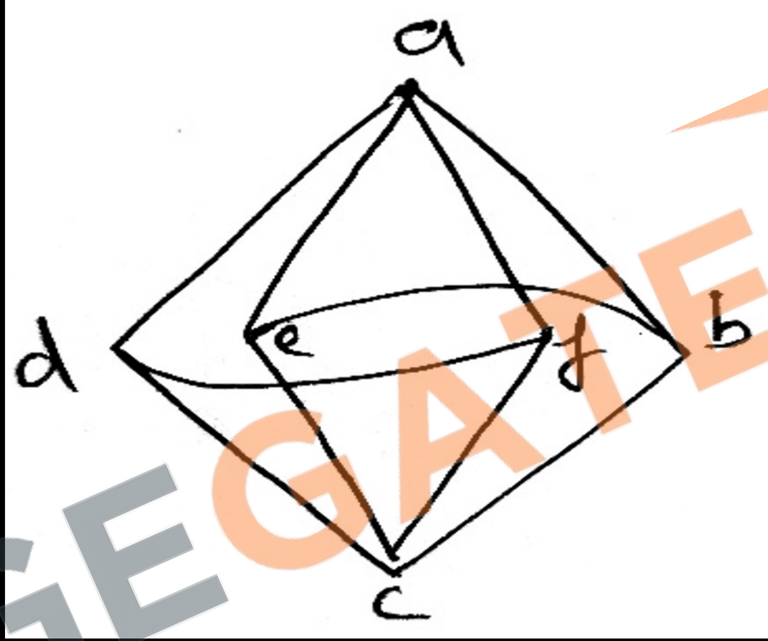

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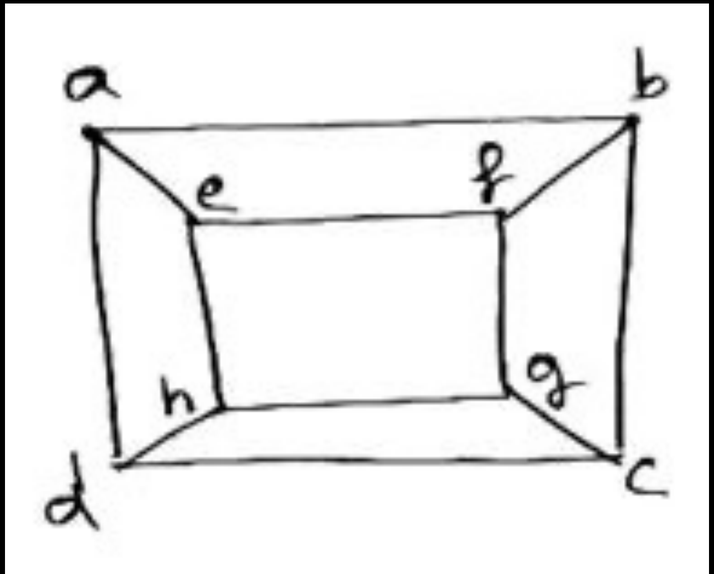
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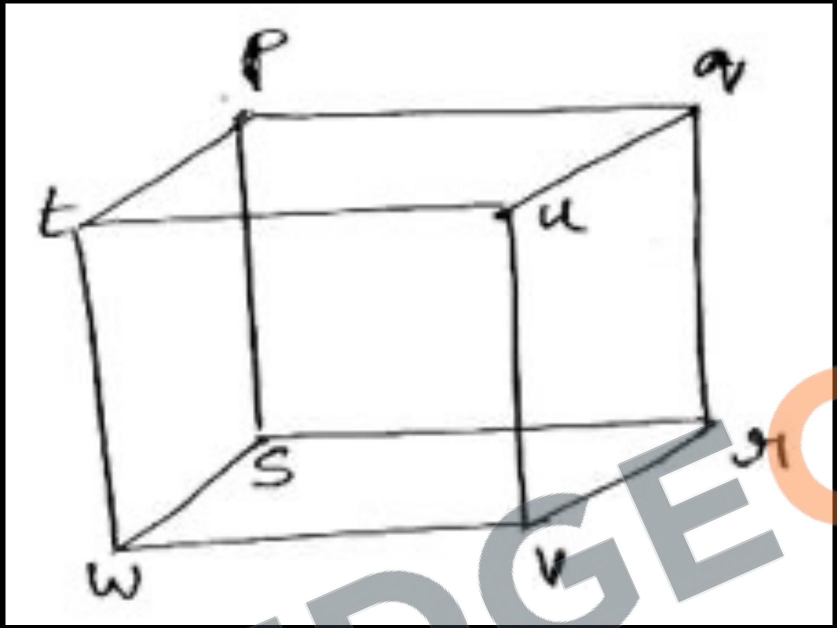
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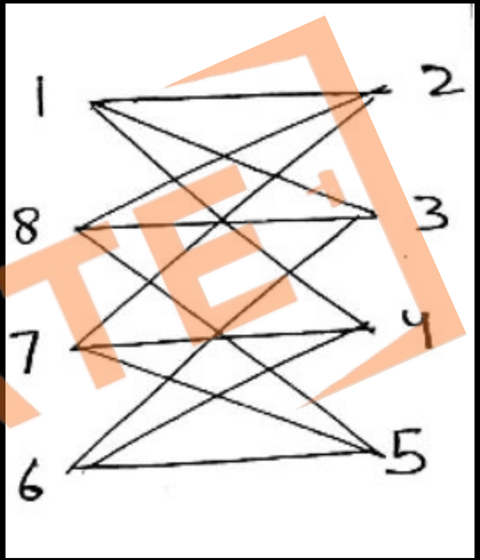
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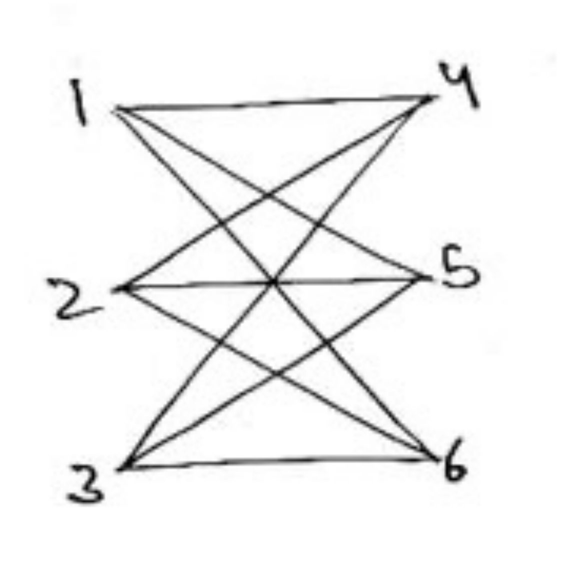
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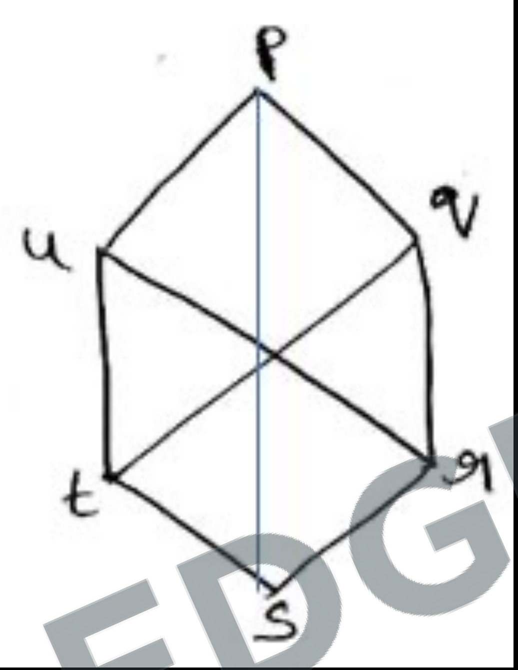
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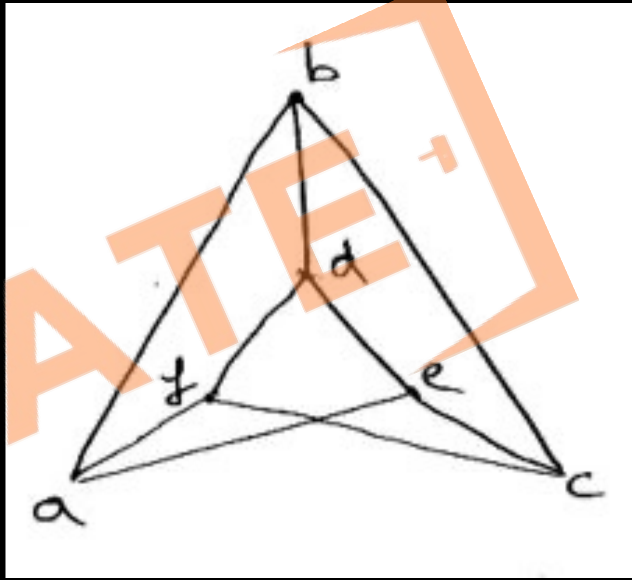
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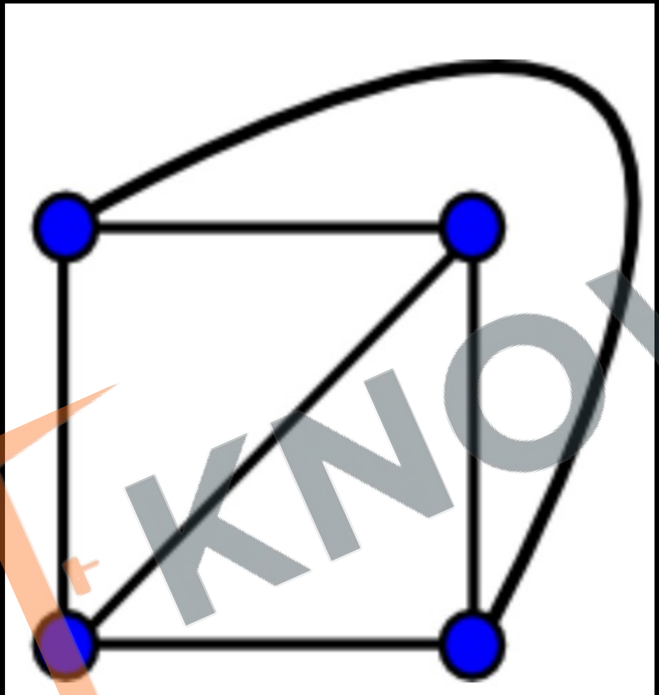
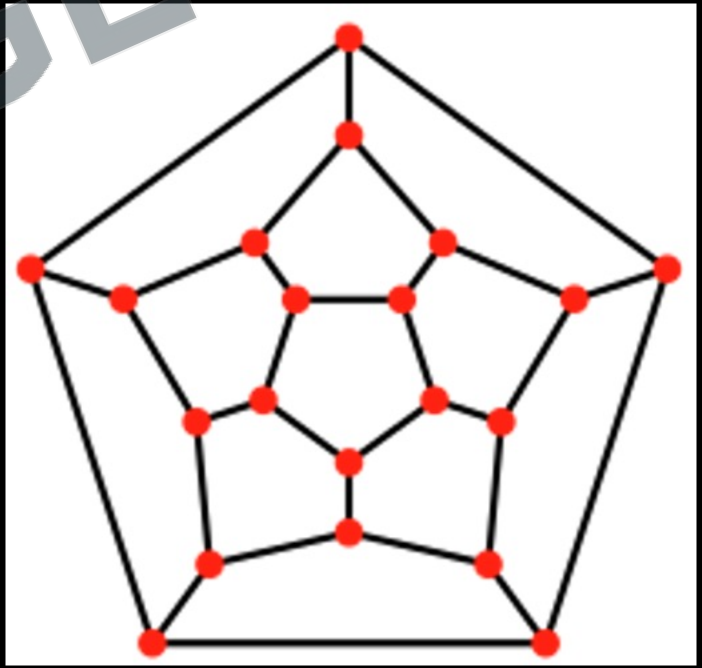
Euler's formula

- A planer graph divides the plane into number or regions (faces, planer embedding), which are further divided into bounded(internal) and unbounded region(external). **Euler's formula** states that if a finite, connected, planar graph with *v* is the number of vertices, *e* is the number of edges and *r* is the number of faces (regions bounded by edges, including the outer, infinitely large region), then r = e – v + 2





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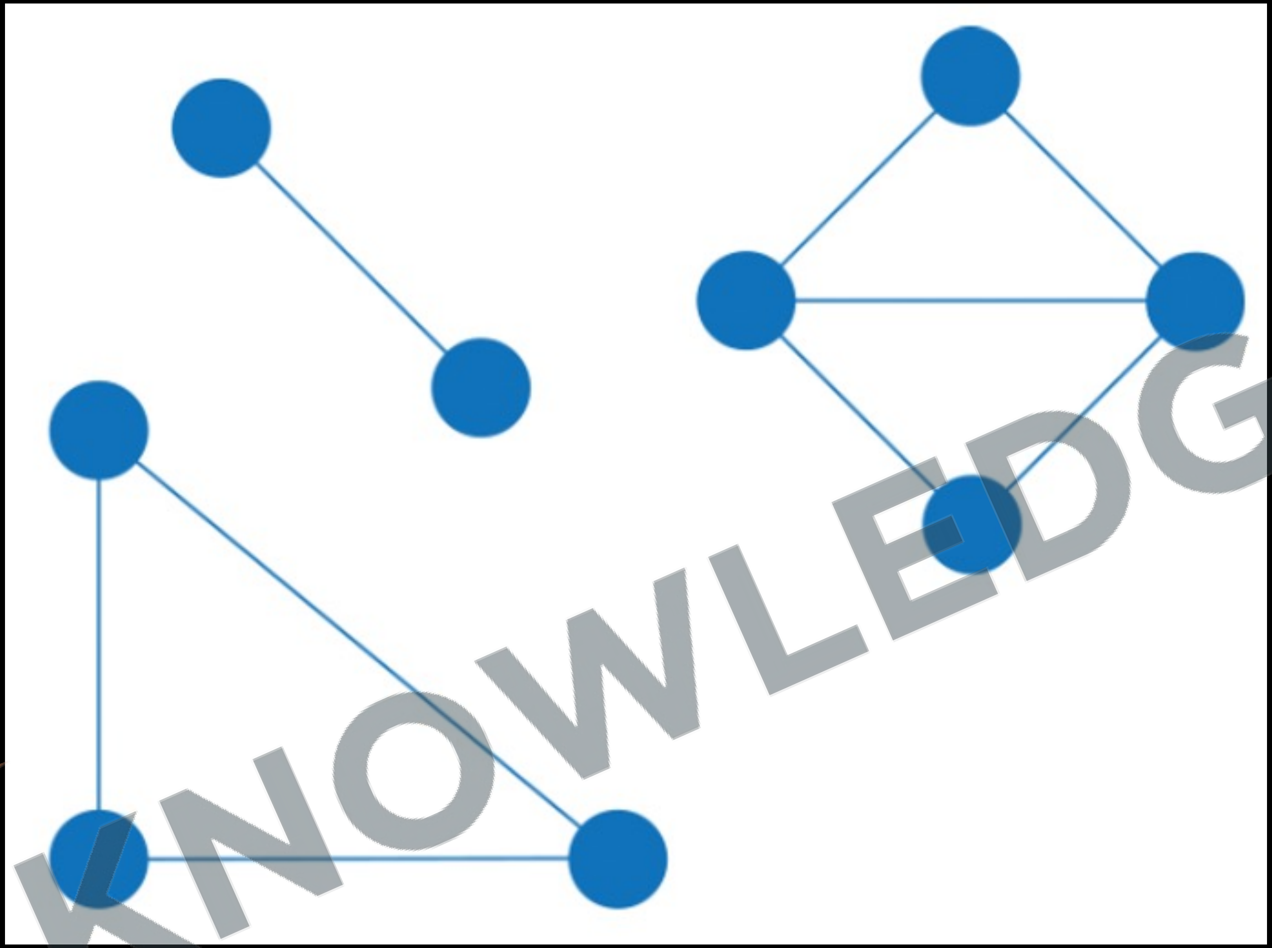
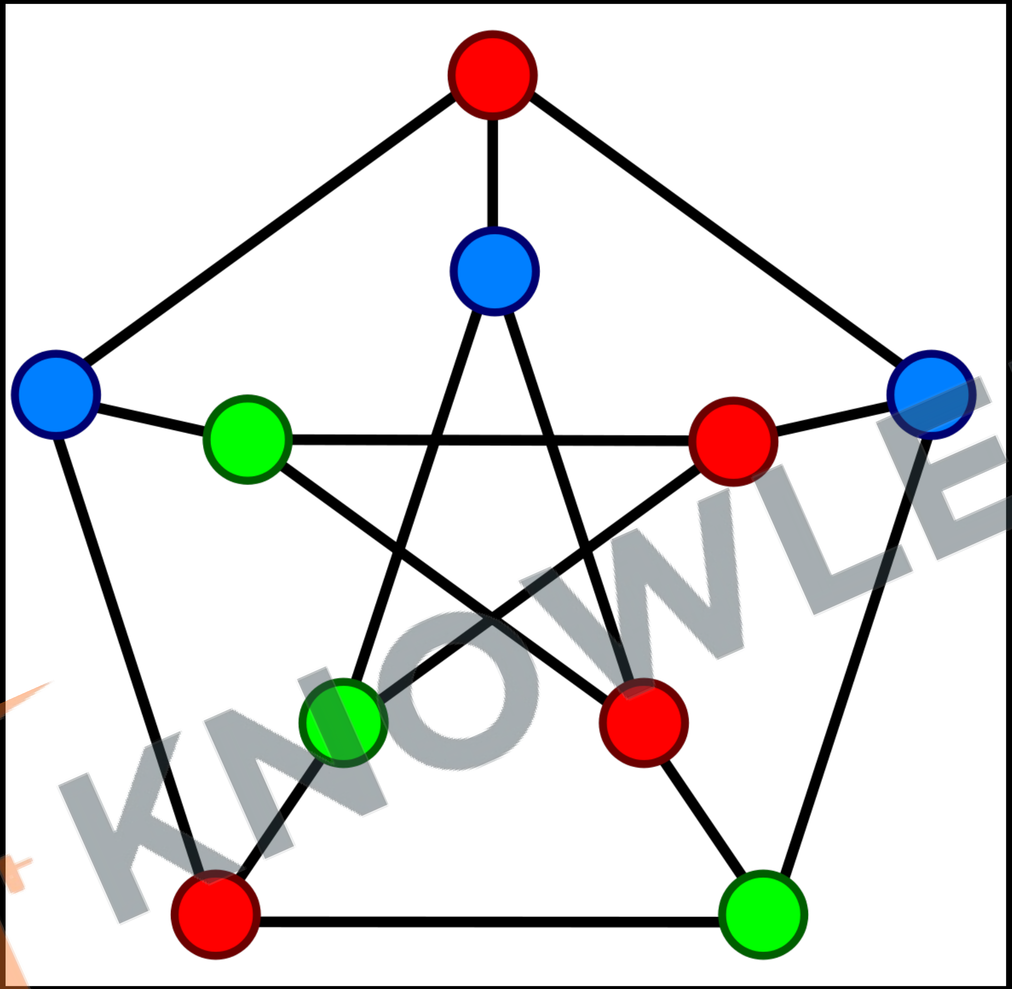
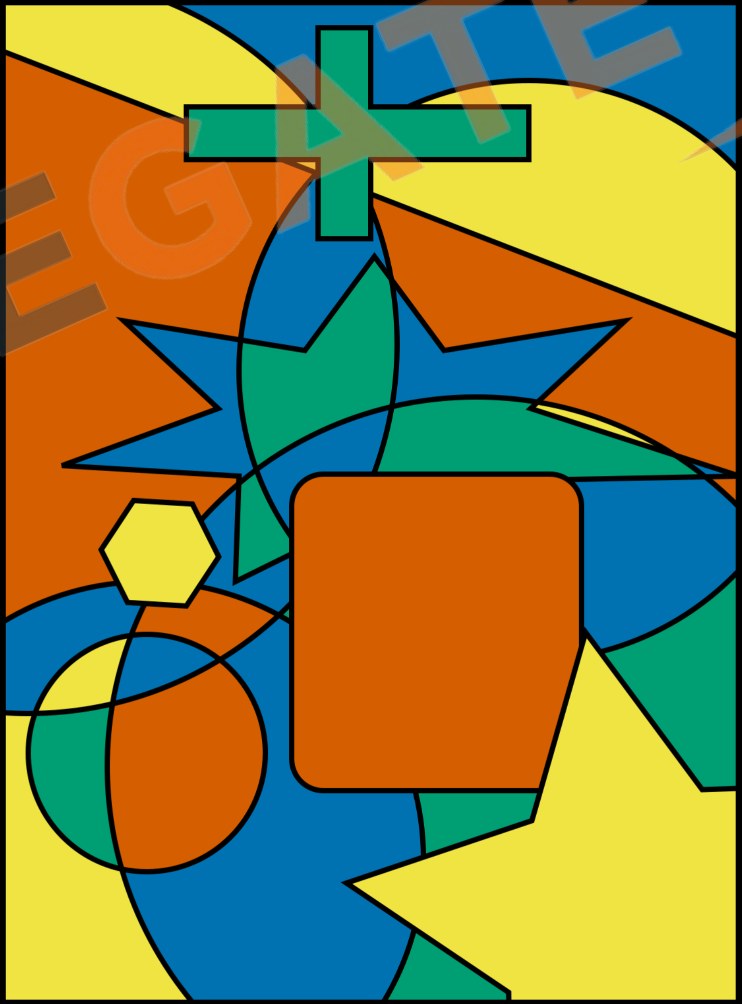
- Euler's formula (Disconnected graph): $V - e + r - k = 1$



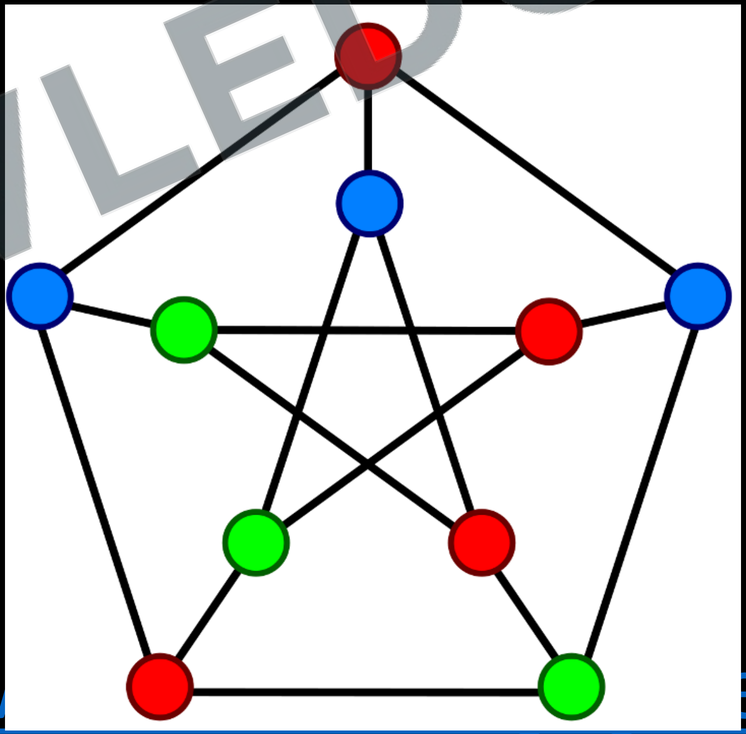
Graph Coloring

- Graph coloring can be of two types vertex coloring and region coloring.





- Associating a color with each vertex of the graph is called vertex coloring.
- **Proper Vertex coloring:** - Associating all the vertex of a graph with colors such that no two adjacent vertices have the same color is called proper vertex coloring.
- **Chromatic number of the graph:** - Minimum number of colors required to do a proper vertex coloring is called the chromatic number of the graph, denoted by $\chi(G)$. the graph is called K-chromatic or K-colorable.



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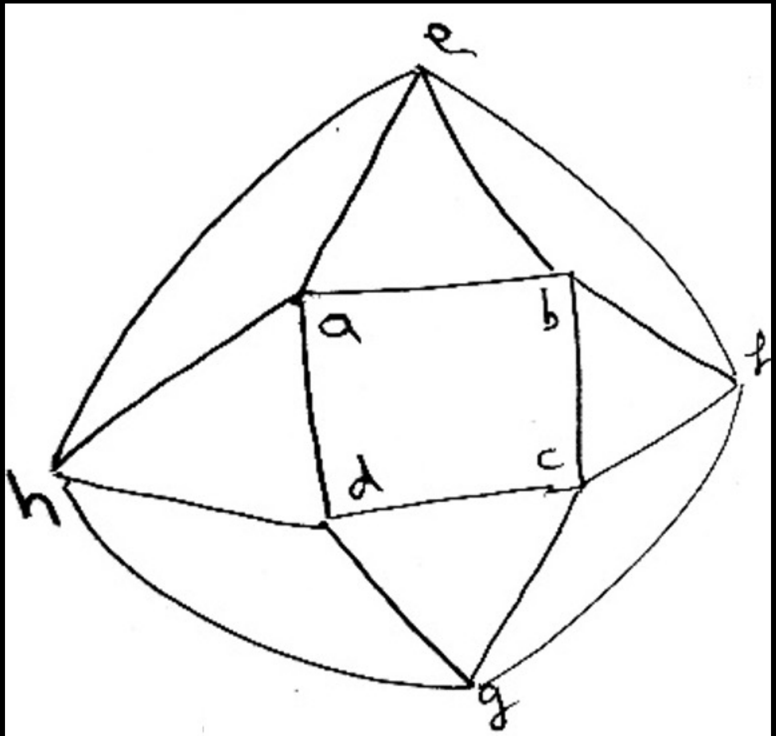
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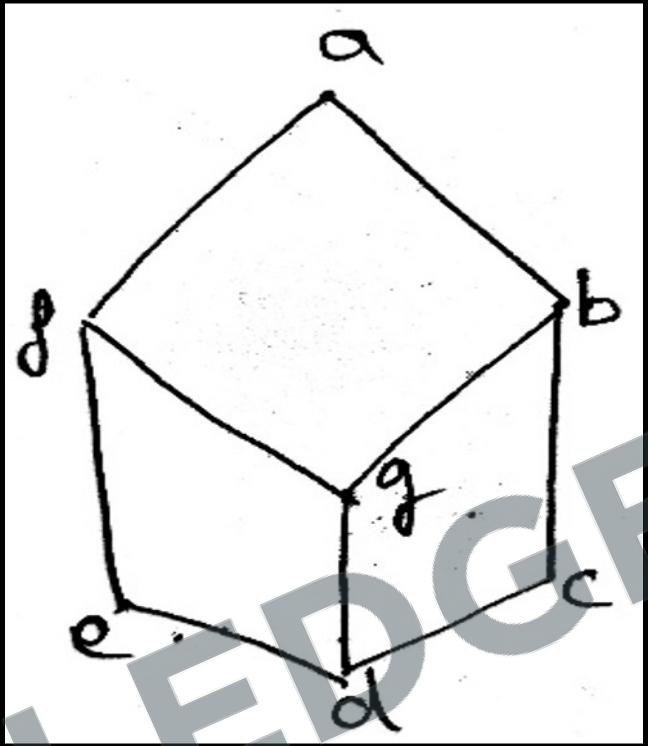
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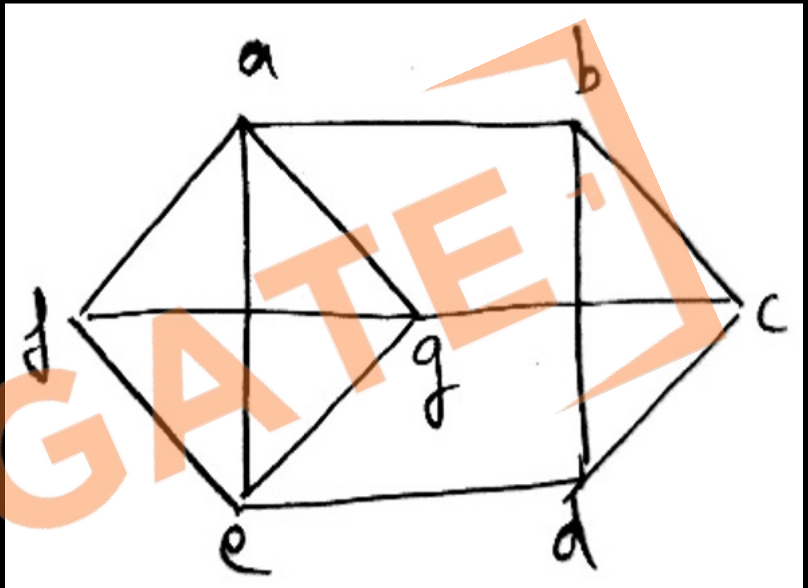
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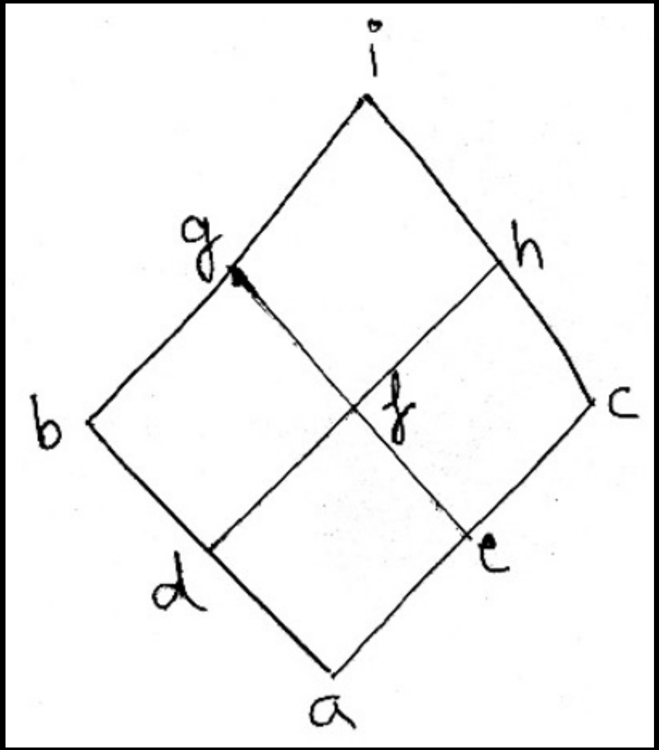
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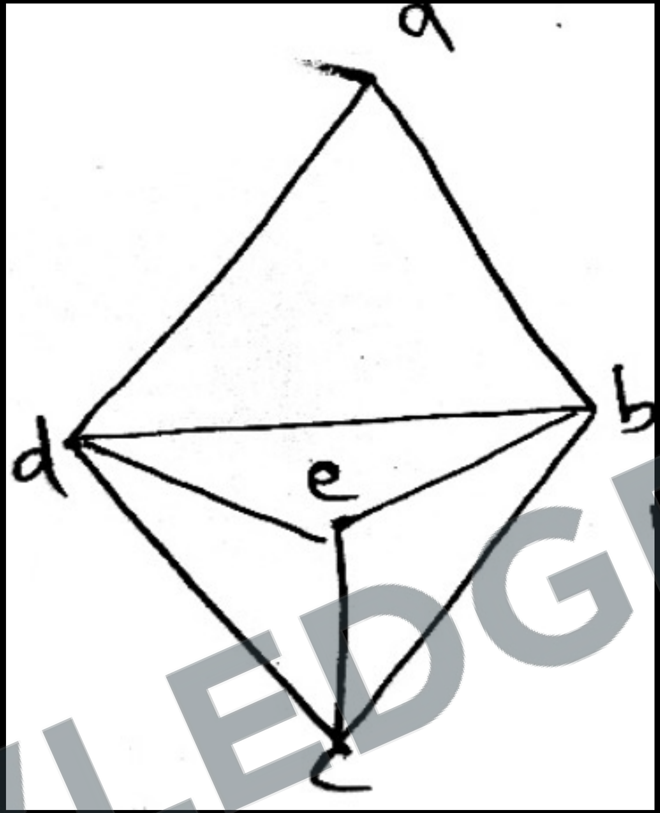
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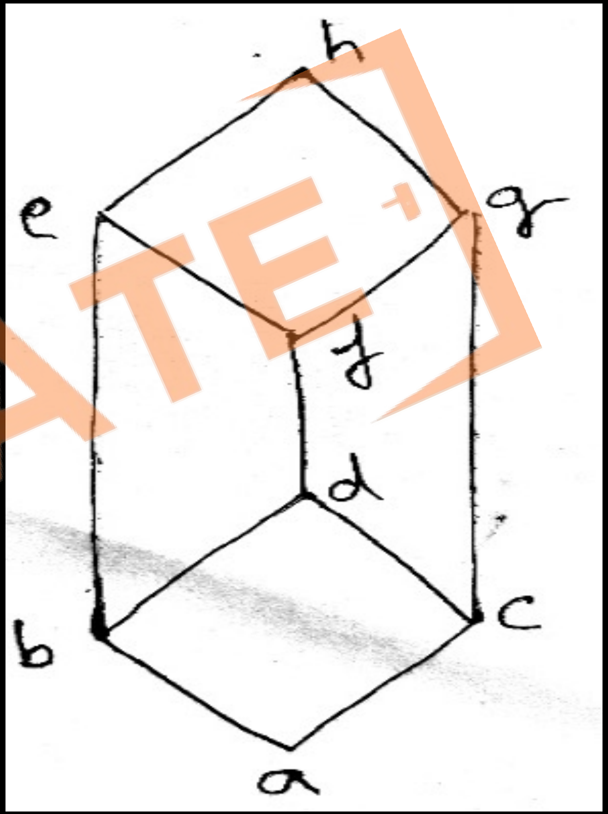
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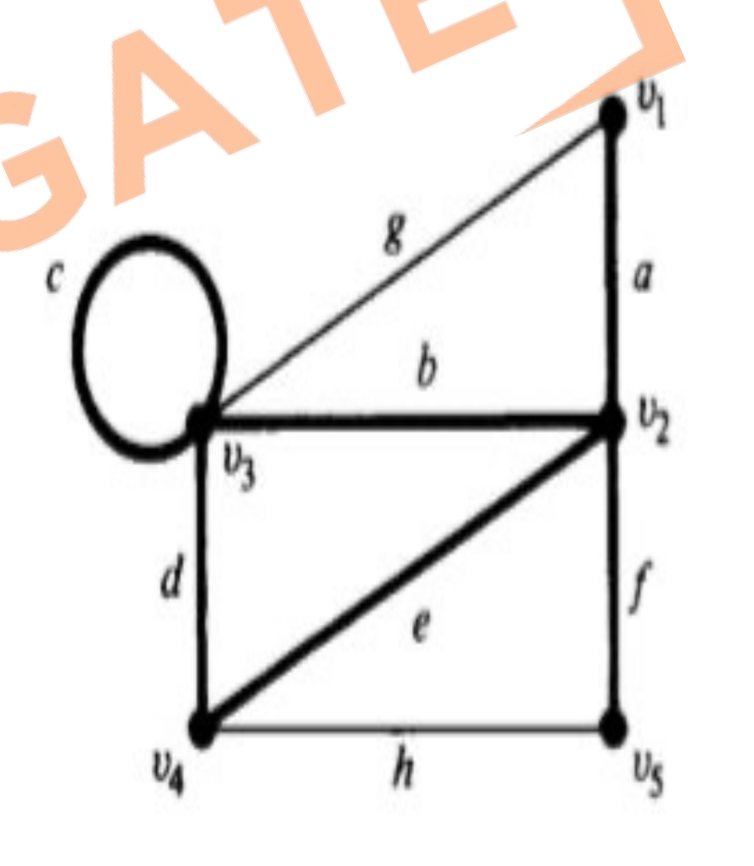
Traversal

- **Walk / Edge Train / Chain**: -A Walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices. Vertex and edge, may appear more than once in a walk.
- It is possible for a walk to begin and end at the same vertex. Such a walk is called a closed walk. A walk that is not closed is called an open walk.
- An open walk in which no vertex appears more than once is called a path (a path does not interact itself). Number of edges in a path is called length of a path.



Open Walk	
Closed Walk	
Path	

- **Connected Graph:** A graph is said to be connected if there is at least one path between every pair of vertices in G.
- A graph with n vertices can be connected with minimum n -1 edges.
- A graph with n vertices will necessary be connected if it has more than (n - 1) (n - 2)/2 edges.

Q Which condition is necessarily for a graph to be connected?

a) A graph with 6 vertices and 10 edges

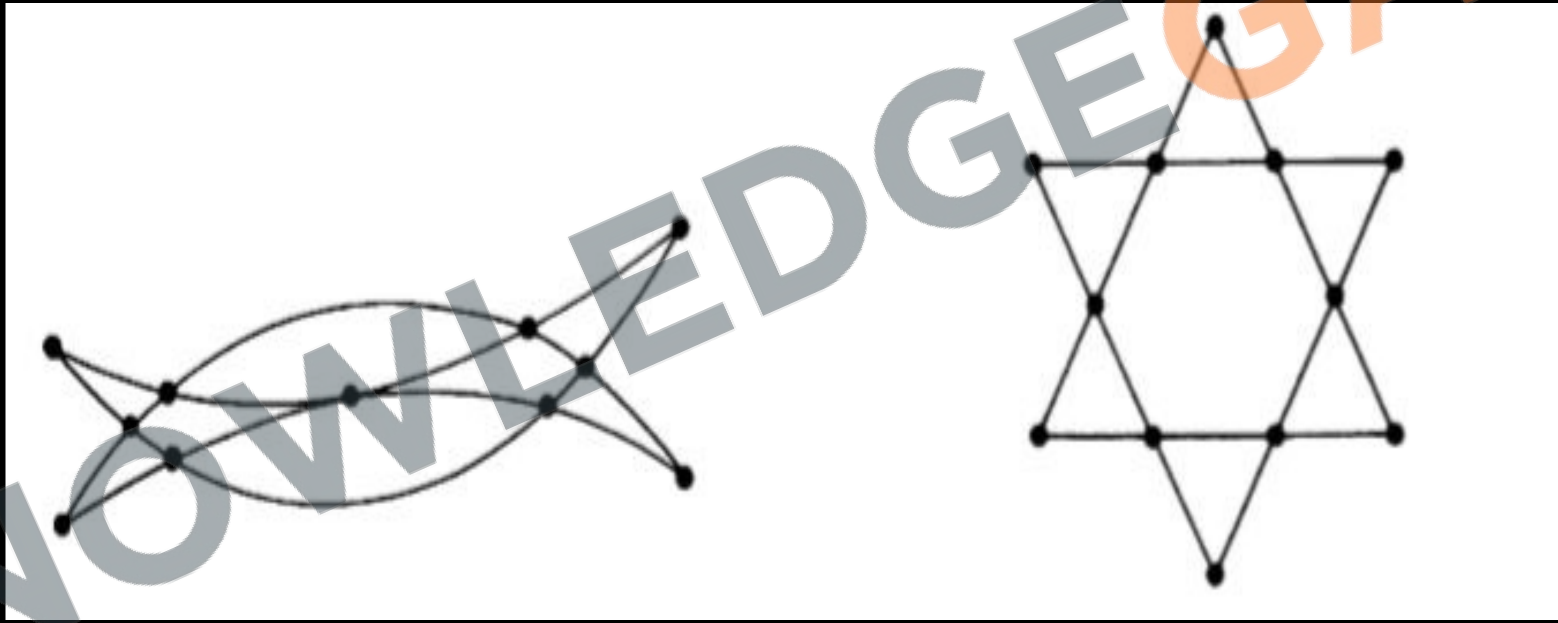
b) A graph with 7 vertices and 14 edges

c) A graph with 8 vertices and 22 edges

d) A graph with 9 vertices and 28 edges

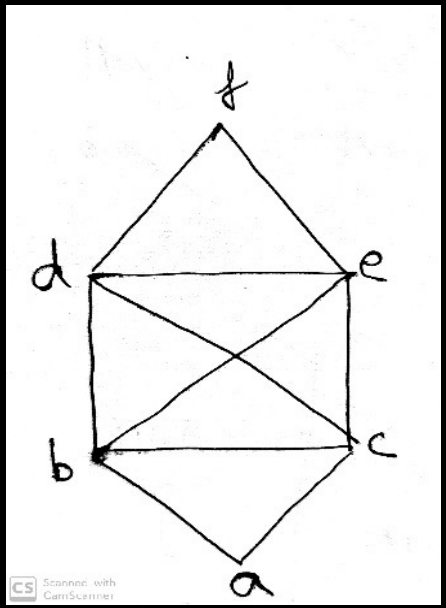
Euler Graph

1. **Euler Graph:** - If some closed walk in a graph contains all the edges of the graph(connected), then the walk is called a Euler line and the graph a Euler Graph.
2. A given connected graph G is a Euler graph if and only if all vertices of G are of even degree.



Hamiltonian

1. **Hamiltonian Graph:** - A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except of course the starting vertex, at which the walk also terminates. A graph containing Hamiltonian circuit is called Hamiltonian graph.
2. Finding weather a graph is Hamiltonian or not is a NPC problem.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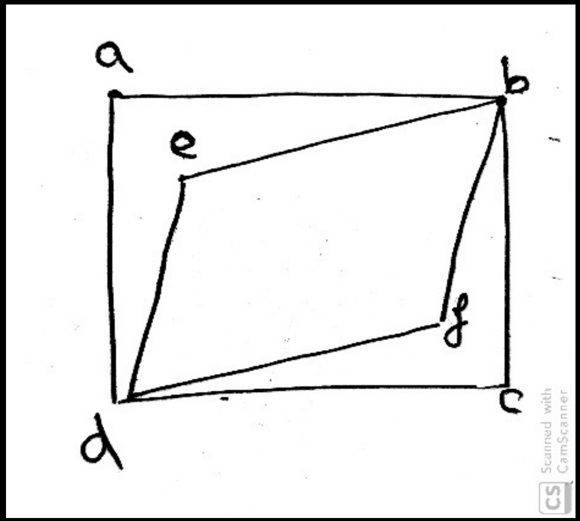
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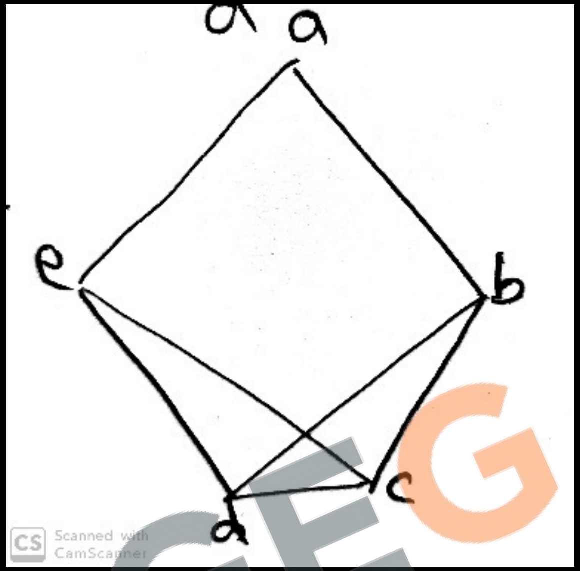
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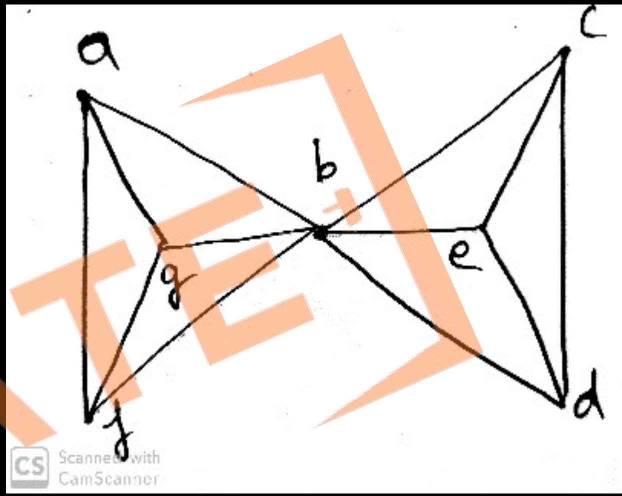
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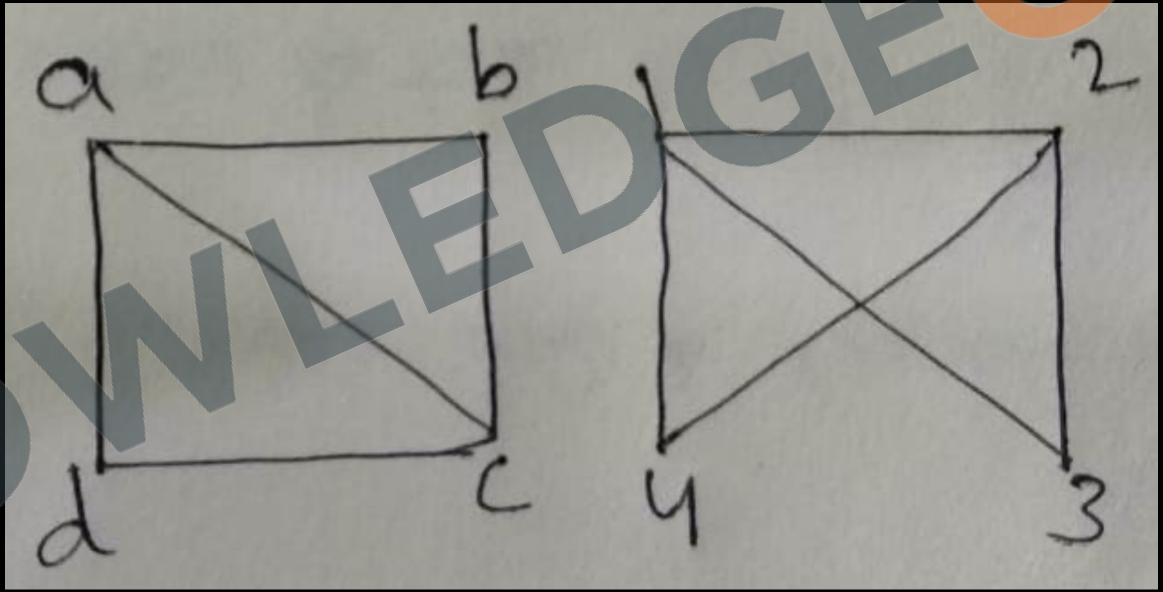
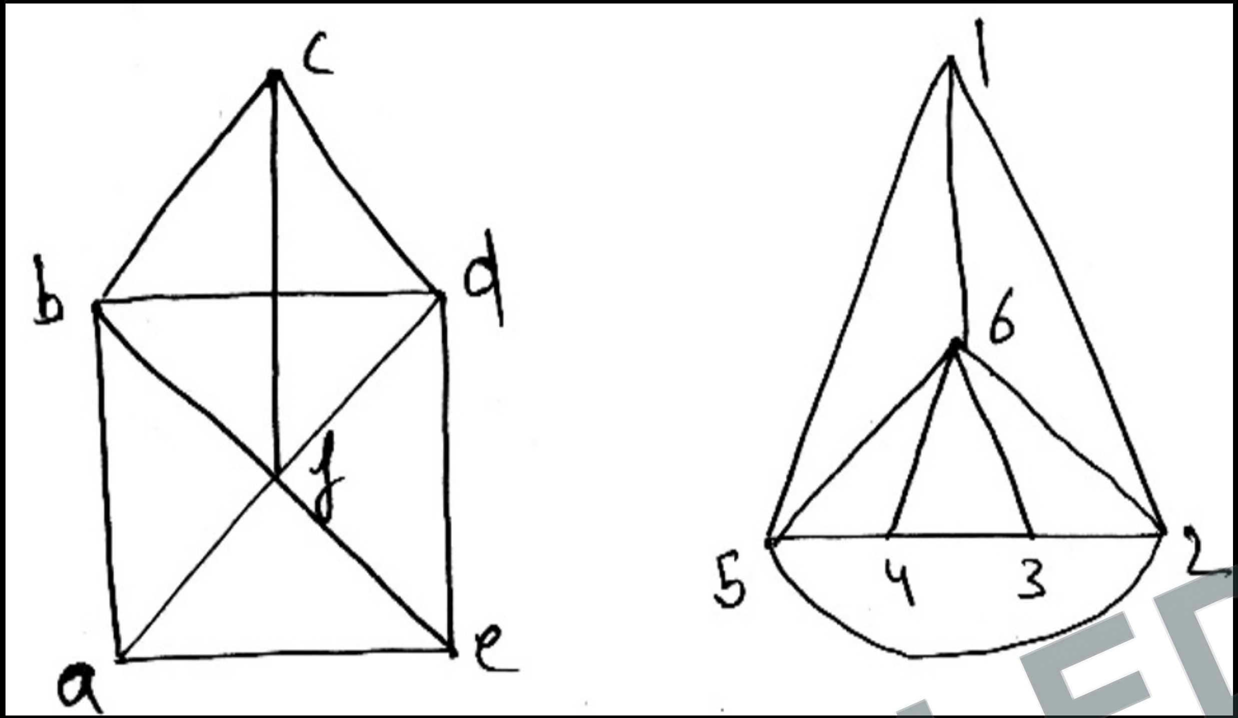
Isomorphism

1. In general, two graphs are said to be isomorphic if they are perhaps the same graphs, but just drawn differently with different names. i.e. two graphs are thought of as isomorphic if they have identical behavior in terms of graph-theoretic properties.



Q How many simple non isomorphic graphs are possible with 3 vertices ?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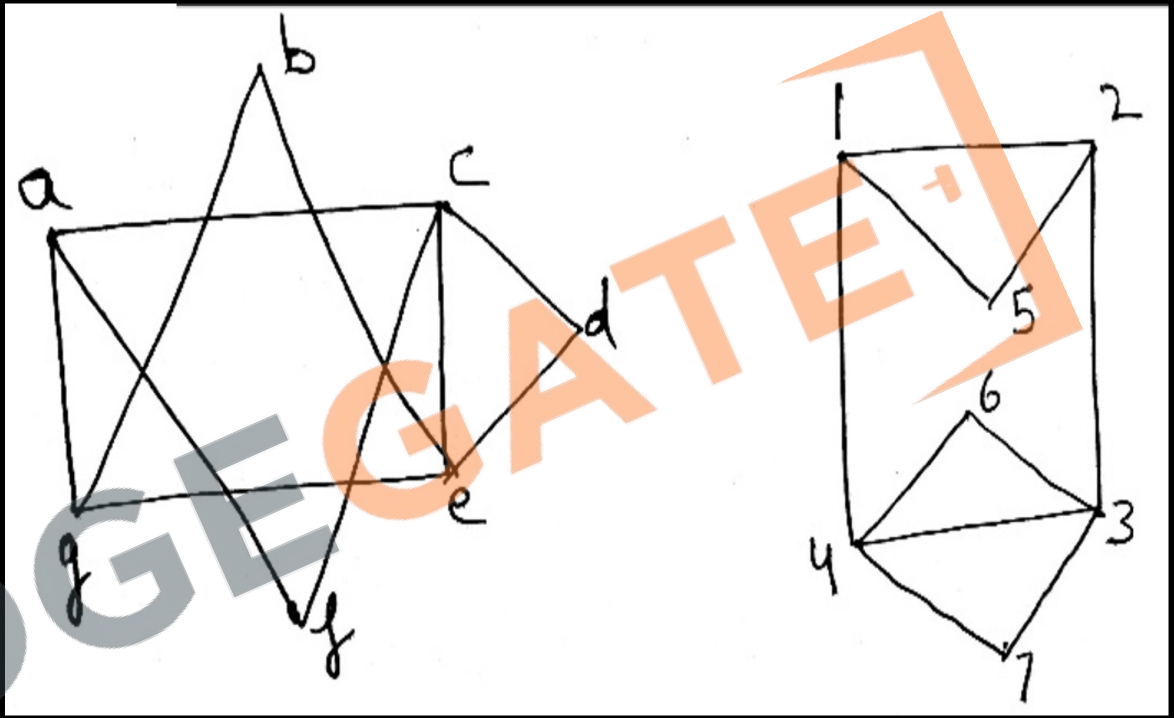
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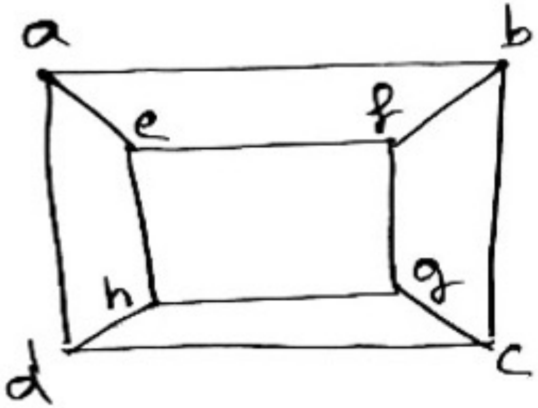
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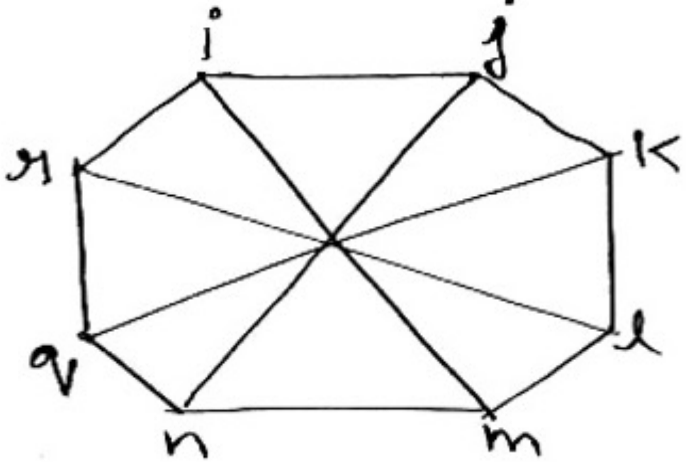
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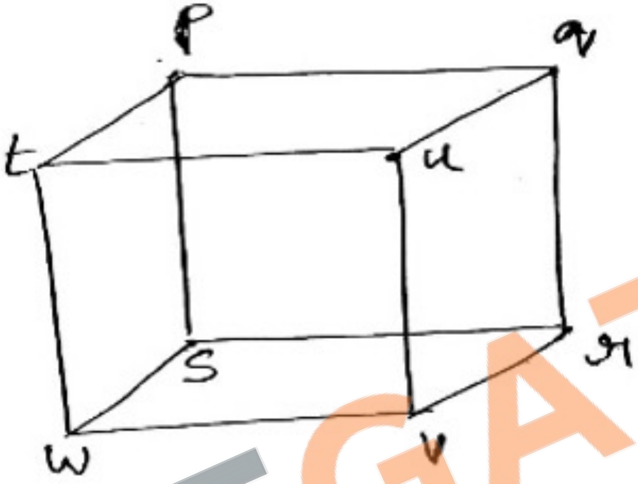
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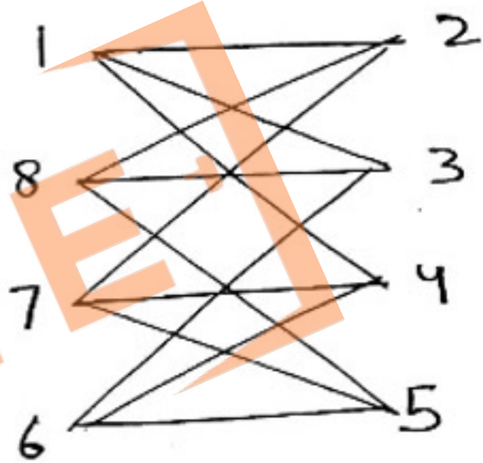
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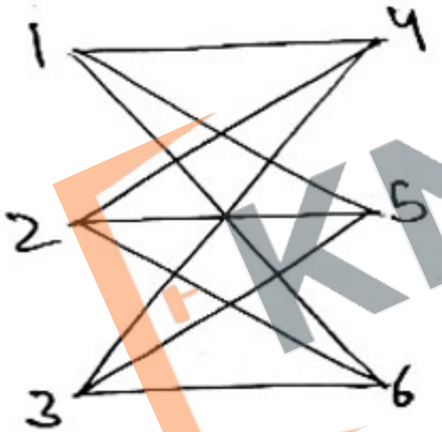
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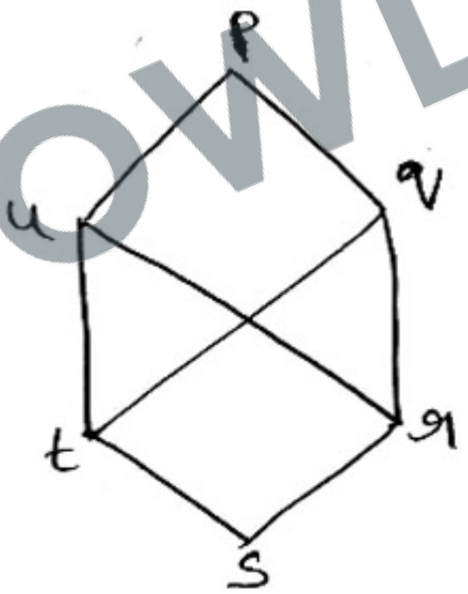
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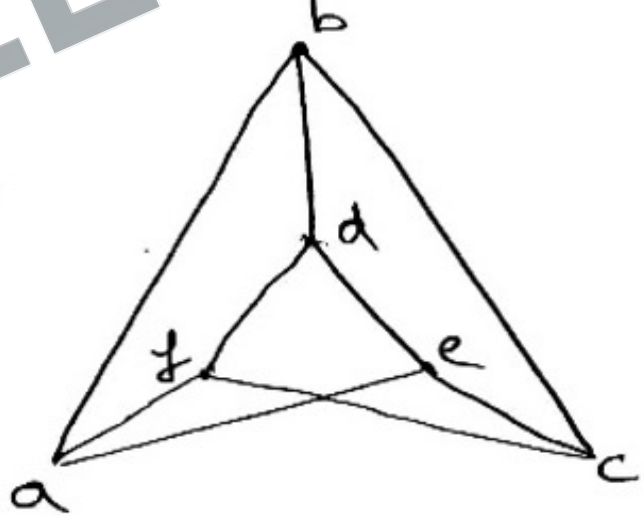
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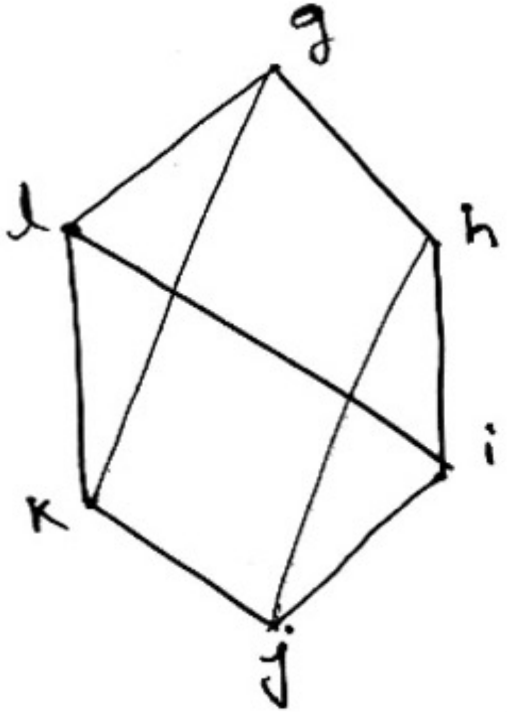
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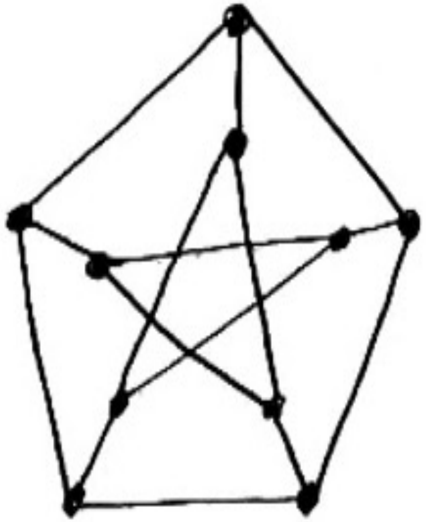
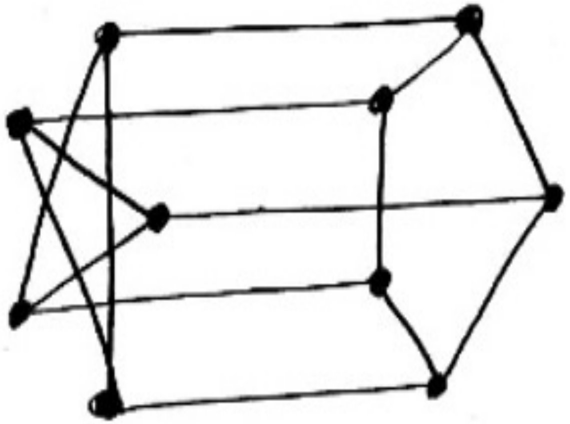
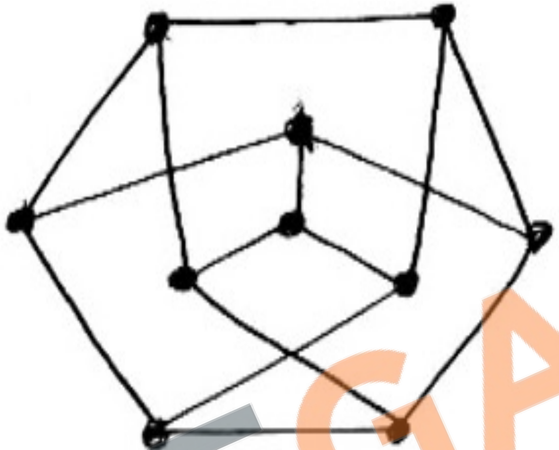
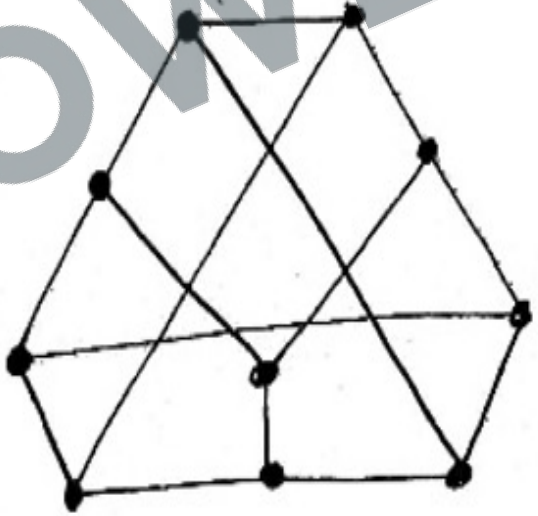
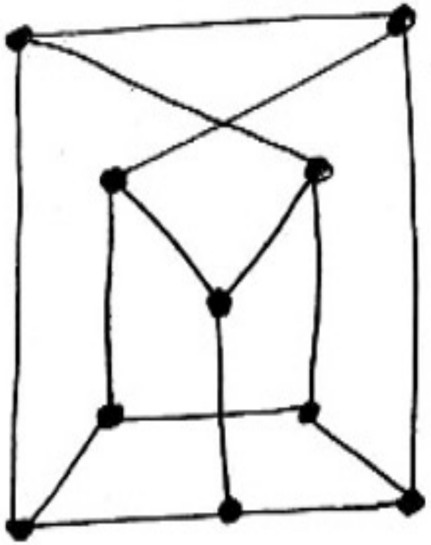
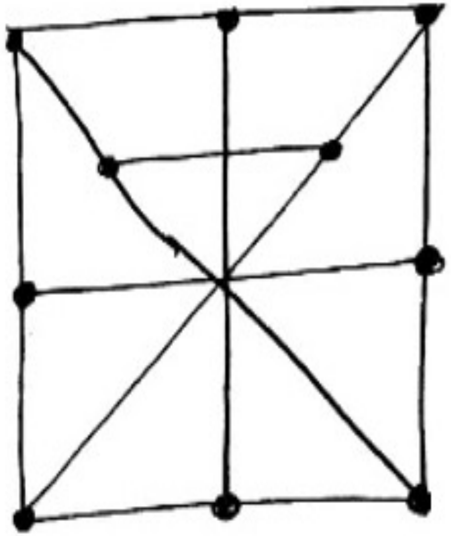












Chapter-8 (Combinatorics)

Permutation

- Permutation refers to the arrangement of a set of items or elements in a specific order. The key points about permutation are:
  - **Order Matters**: The arrangement is considered different if the order of items is different.
  - **Counting Permutations**: To find the number of permutations of 'n' items taken 'r' at a time, we use the formula $^{n}p_{m}$ = P (n, m).

Examples:

- **Simple Example**:
  - Items: A, B
  - Permutations: AB, BA
  - Total: 2 permutations

- **Numerical Example**:
  - Items: 1, 2, 3
  - Permutations: 123, 132, 213, 231, 312, 321
  - Total: 6 permutations ($3! = 3 \times 2 \times 1 = 6$)

- **Real-world Example**:
  - Scenario: Arranging books on a shelf.
  - Books: Physics (P), Chemistry (C), Mathematics (M)
  - Permutations: PCM, PMC, CMP, CPM, MPC, MCP
  - Total: 6 arrangements

Combination

- Combination refers to the selection of items from a larger set where the order of selection does not matter. The key points about combination are:
  - **Order Doesn't Matter**: Unlike permutations, the arrangement or order of the selected items is irrelevant in combinations.
  - **Counting Combinations**: To find the number of combinations of 'n' items taken 'r' at a time, we use the formula ${}^{n}C_{r} = (n!) / ((n-r)!n!)$.

- **Simple Example**:
  - Items: A, B, C
  - Selecting 2 items at a time.
  - Combinations: AB, AC, BC
  - Total: 3 combinations

- **Numerical Example**:
  - Items: 1, 2, 3, 4
  - Selecting 2 items at a time.
  - Combinations: 12, 13, 14, 23, 24, 34
  - Total: 6 combinations ($\binom{4}{2}=\frac{4!}{2!(4-2)!}=6$ $4C2=\frac{4!}{2!(4-2)!}=6$)

- **Real-world Example**:
  - Scenario: Choosing 2 fruits from a basket containing an Apple (A), Banana (B), and Cherry (C).
  - Combinations: AB, AC, BC
  - Total: 3 ways to choose 2 fruits out of 3.

Q A collection of 10 electric bulbs contain 3 defective ones?

- In how many ways can a sample of four bulbs be selected?

$^{10}C_4$

- In how many ways can a sample of 4 bulbs be selected which Contain 2 good bulbs and 2 defective ones ?

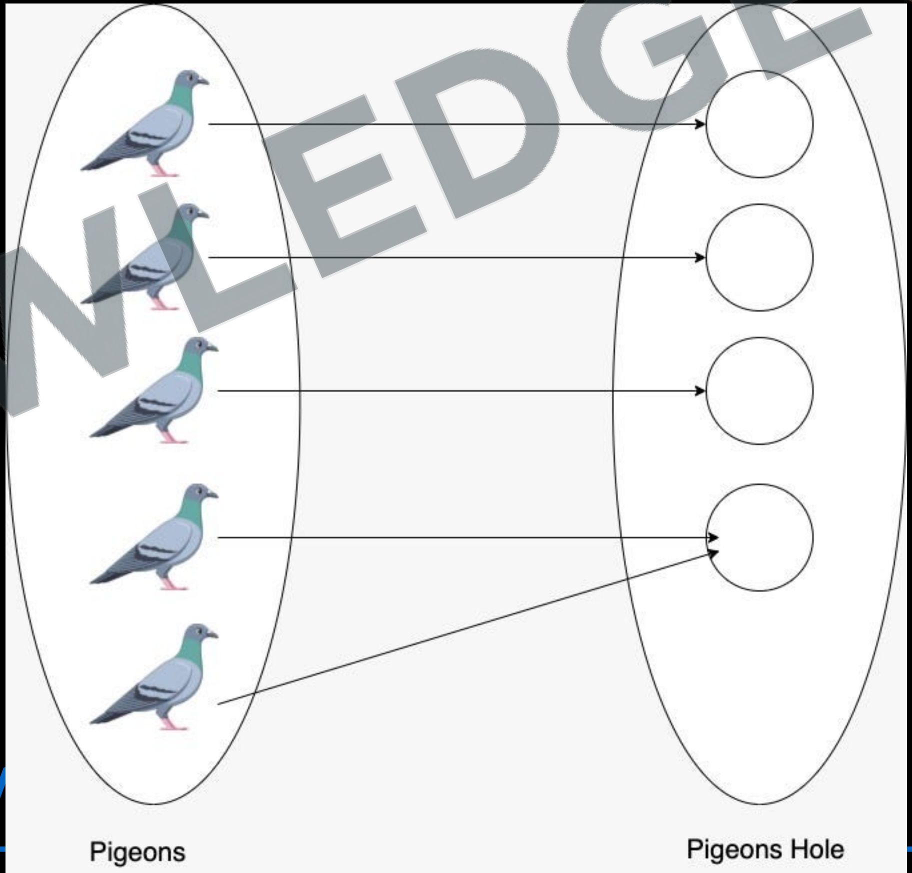
$^{7}C_2 + {}^{3}C_2$

- In how many ways can a sample of 4 bullbs be selected so that either the sample contain 3 good ones and 1 defectives ones or 1 good and 3 defectives ones ?

$^{7}C_3 \times {}^{3}C_1 + {}^{7}C_1 \times {}^{3}C_8$

Pigeonhole Principle

- It states that if more items (pigeons) are put into fewer containers (pigeonholes) than there are items, then at least one container must contain more than one item. Key points include:
  - **Basic Idea**: If 'n' items are distributed among 'm' containers, and if $n>m$, then at least one container has more than one item.



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- **Simple Example**:
  - Socks: 10 pairs of socks (20 individual socks) and 19 drawers.
  - According to the Pigeonhole Principle, since there are more socks than drawers, at least one drawer must contain more than one sock.

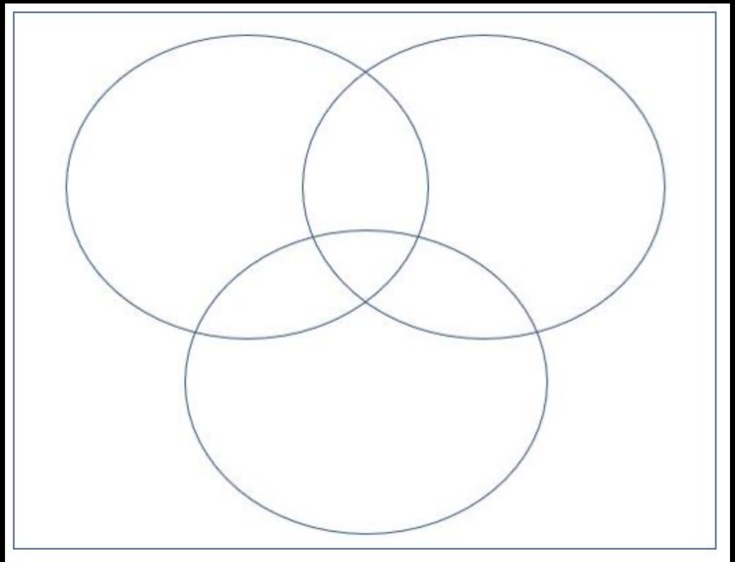
- **Real-world Example**:
  - Scenario: A classroom with 30 students.
  - If you want to prove that at least two students have their birthdays in the same month, you consider the months (12 pigeonholes) and the students (30 pigeons).
  - According to the Pigeonhole Principle, at least one month (pigeonhole) will have more than $\lceil 30/12 \rceil = 3$ students (pigeons) sharing their birthdays.

No of onto function possible from A to B

$= n^m - {}^nc_1(n-1)^m + {}^nc_2(n-2)^m - {}^nc_3(n-3)^m + \text{----------} + (-1)^n \, {}^nc_{n-1} \, 1^m$

Q In a class of 200 students, 125 students have taken Programming Language course, 85 students have taken Data Structures course, 65 students have taken Computer Organization course; 50 students have taken both Programming Language and Data Structures, 35 students have taken both Data Structures and Computer Organization; 30 students have taken both Programming Language and Computer Organization, 15 students have taken all the three courses. How many students have not taken any of the three courses?

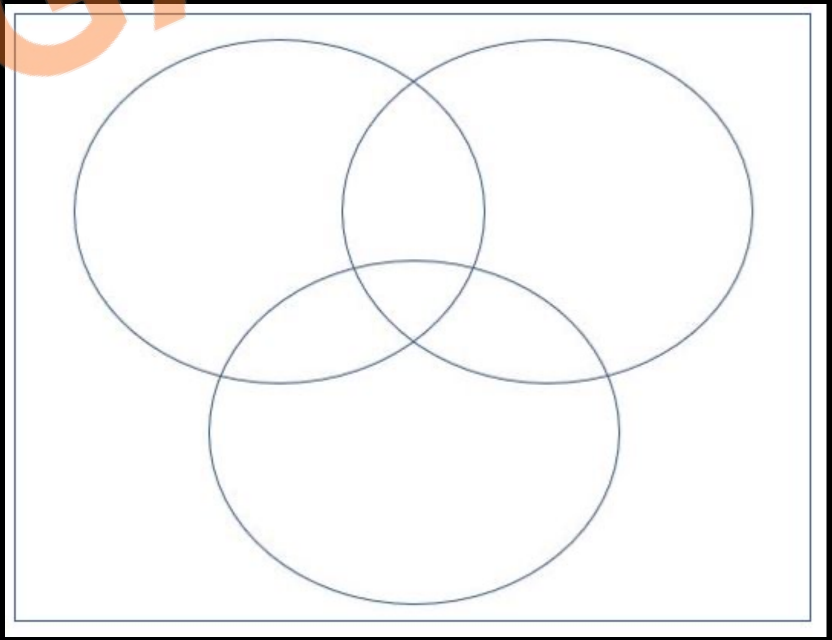
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



125+85+65-50–35–30+15

Q The number of integers between 1 and 500 (both inclusive) that are divisible by 3 or 5 or 7 is _______.

$A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$



166 + 100 + 71 -33-14-23+4

Q In a college, there are three student clubs, Sixty students are only in the Drama club, 80 students are only in the Dance club, 30 students are only in Maths club, 40 students are in both Drama and Dance clubs, 12 students are in both Dance and Maths clubs, 7 students are in both Drama and Maths clubs, and 2 students are in all clubs. If 75% of the students in the college are not in any of these clubs, then the total number of students in the college is ______________.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$